LAHORE UNIVERSITY OF MANAGEMENT SCIENCES
Department of Electrical Engineering

AI 501 Mathematics for Artificial Intelligence
Quiz 2 Solutions

Name: ______________________

Campus ID: ______________________

Total Marks: 10
Time Duration: 15 minutes

Question 1 (4 marks)

You are working on a project to analyze customer purchasing behavior in an online retail store. You are given three customers' purchase history represented as vectors in a three-dimensional space, where each dimension corresponds to the number of purchases in three different product categories: electronics, clothing, and groceries.
The purchase history vectors for three customers are:

$$\mathbf{v}_A = \begin{pmatrix} 15 \\ 7 \\ 3 \end{pmatrix}, \mathbf{v}_B = \begin{pmatrix} 6 \\ 12 \\ 7 \end{pmatrix}, \mathbf{v}_C = \begin{pmatrix} 14 \\ 2 \\ 9 \end{pmatrix}$$

(a) How are the purchasing habits of customer A correlated with that of customer C?

Solution: The Pearson correlation coefficient between two vectors is calculated by:

$$\rho = \frac{\langle (\mathbf{v}_A - \bar{A}), (\mathbf{v}_C - \bar{C} \rangle}{\|\mathbf{v}_A - \bar{A}\| \|\mathbf{v}_C - \bar{C}\|}$$

where $\bar{A}$ and $\bar{C}$ are the means of vectors $\mathbf{v}_A$ and $\mathbf{v}_C$, respectively.
First, we compute the means of $\mathbf{v}_A$ and $\mathbf{v}_C$:

$$\bar{A} = \frac{15 + 7 + 3}{3} = \frac{25}{3} \approx 8.33$$

$$\bar{C} = \frac{14 + 2 + 9}{3} = \frac{25}{3} \approx 8.33$$

Next, we center the vectors by subtracting their means:

$$\mathbf{v}'_A = \begin{pmatrix} 15 - 8.33 \\ 7 - 8.33 \\ 3 - 8.33 \end{pmatrix} = \begin{pmatrix} 6.67 \\ -1.33 \\ -5.33 \end{pmatrix}$$

$$\mathbf{v}'_C = \begin{pmatrix} 14 - 8.33 \\ 2 - 8.33 \\ 9 - 8.33 \end{pmatrix} = \begin{pmatrix} 5.67 \\ -6.33 \\ 0.67 \end{pmatrix}$$

Now, we compute the dot product of the centered vectors:

$$\langle \mathbf{v}'_A, \mathbf{v}'_C \rangle = (6.67 \times 5.67) + (-1.33 \times -6.33) + (-5.33 \times 0.67)$$
$$= 37.82 + 8.42 - 3.57 = 42.67$$

Next, we compute the norms of the centered vectors:

$$\|\mathbf{v}'_A\| = \sqrt{6.67^2 + (-1.33)^2 + (-5.33)^2} = \sqrt{44.49 + 1.77 + 28.41} = \sqrt{74.67} \approx 8.64$$
$$\|\mathbf{v}'_c\| = \sqrt{5.67^2 + (-6.33)^2 + 0.67^2} = \sqrt{32.14 + 40.07 + 0.45} = \sqrt{72.66} \approx 8.52$$

Finally, we compute the Pearson correlation coefficient:

$$\rho = \frac{42.67}{8.64 \times 8.52} = \frac{42.67}{73.58} \approx 0.58$$

Thus, the Pearson correlation coefficient between the purchasing habits of Customer A and Customer C is approximately $\rho = 0.58$, indicating a moderate positive correlation.

(b) Another task on the project requires you to form an "average" customer profile from the purchase histories of Customers A, B and C. You are told that the combination must be convex, and that the weight of the purchasing history of Customer A $w_a = 0.3$. Choosing appropriate weights w_b and w_c, compute the "average" customer profile. We can have multiple answers to this problem.

Solution: The "average" customer profile, $\mathbf{v}_{ave}$ can be expressed as:

$$\mathbf{v}_{ave} = w_a\mathbf{v}_a + w_b\mathbf{v}_b + w_c\mathbf{v}_c.$$

Also note that for the combination to be convex, the following condition must hold:

$$w_a + w_b + w_c = 1 =, \text{ and } w_a, w_b, w_c \leq 1.$$

There are many possible different answers (infinitely many in fact). We choose our weights as $w_2 = 0.4$ and $w_3 = 0.3$. Then, the average customer profile is:

$$\mathbf{v}_a ve = 0.3\mathbf{v}_A + 0.4\mathbf{v}_B + 0.3\mathbf{v}_C$$

plugging in the values, we obtain:

$$\mathbf{v}_a ve = \begin{pmatrix} 11.1 \\ 7.5 \\ 6.4 \end{pmatrix}$$

Question 2 (6 marks)

Suppose we have the following three vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

(a) Are these vectors mutually orthogonal to each other?

Solution: To determine if the vectors are orthogonal, we need to check if the dot product between any pair of vectors is zero. The dot product of two vectors **a** and **b** is given by:

$$\langle \mathbf{a}, \mathbf{b} \rangle = a_1b_1 + a_2b_2 + a_3b_3$$

Let's compute the dot products for each pair:
$\mathbf{v}_1 \cdot \mathbf{v}_2 = (1)(0) + (0)(2) + (2)(0) = 0$
$\mathbf{v}_1 \cdot \mathbf{v}_3 = (1)(2) + (0)(0) + (2)(-1) = 2 - 2 = 0$
$\mathbf{v}_2 \cdot \mathbf{v}_3 = (0)(2) + (2)(0) + (0)(-1) = 0$
Since the dot product between each pair of vectors is zero, the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are orthogonal to each other.

(b) Is this set of vectors linearly dependent?

Solution: To check if the set of vectors is linearly dependent, we need to determine if any vector can be written as a linear combination of the others. Since the vectors are orthogonal, they are linearly independent. Therefore, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

(c) Do these vectors span $\mathbb{R}^3$?

Solution: In $\mathbb{R}^3$, a set of three linearly independent vectors spans the entire space. Since the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent, they span $\mathbb{R}^3$.

(d) Compute the normalized version of these vectors such that each vector has a unit 2-norm. Express the following vector as a linear combination of the normalized vectors:

$$\mathbf{x} = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}$$

Solution: To normalize the vectors, we divide each vector by its magnitude (2-norm). The magnitude of a vector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ is given by:

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Let's compute the magnitudes:

$$\|\mathbf{v}_1\| = \sqrt{1^2 + 0^2 + 2^2} = \sqrt{1+4} = \sqrt{5}$$
$$\|\mathbf{v}_2\| = \sqrt{0^2 + 2^2 + 0^2} = \sqrt{4} = 2$$
$$\|\mathbf{v}_3\| = \sqrt{2^2 + 0^2 + (-1)^2} = \sqrt{4+1} = \sqrt{5}$$

Now, normalize the vectors by dividing each by its magnitude:

$$\hat{\mathbf{v}}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$\hat{\mathbf{v}}_2 = \frac{1}{2} \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\hat{\mathbf{v}}_3 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ 0 \\ \frac{-1}{\sqrt{5}} \end{pmatrix}$$

Next, we express the vector $\mathbf{x} = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}$ as a linear combination of the normalized vectors $\hat{\mathbf{v}}_1$, $\hat{\mathbf{v}}_2$, and $\hat{\mathbf{v}}_3$. We want to find scalars c_1, c_2, and c_3 such that:

$$\mathbf{x} = c_1\hat{\mathbf{v}}_1 + c_2\hat{\mathbf{v}}_2 + c_3\hat{\mathbf{v}}_3$$

Substituting the normalized vectors, we have:

$$\mathbf{x} = c_1 \begin{pmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{2}{\sqrt{5}} \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} \frac{2}{\sqrt{5}} \\ 0 \\ \frac{-1}{\sqrt{5}} \end{pmatrix}$$

This gives the system of equations:
1. $c_1 \frac{1}{\sqrt{5}} + c_3 \frac{2}{\sqrt{5}} = 5$ 2. $c_2 = 5$ 3. $c_1 \frac{2}{\sqrt{5}} + c_3 \frac{-1}{\sqrt{5}} = 5$
From Equation (2), we know $c_2 = 5$. Now, solving Equations (1) and (3):
Multiplying both equations by $\sqrt{5}$ to simplify:

$$c_1 + 2c_3 = 5\sqrt{5} \quad (4)$$
$$2c_1 - c_3 = 5\sqrt{5} \quad (5)$$

From Equation (5), solve for c_3:

$$c_3 = 2c_1 - 5\sqrt{5}$$

Substitute into Equation (4):

$$c_1 + 2(2c_1 - 5\sqrt{5}) = 5\sqrt{5}$$
$$c_1 + 4c_1 - 10\sqrt{5} = 5\sqrt{5}$$
$$5c_1 = 15\sqrt{5}$$
$$c_1 = 3\sqrt{5}$$

Now, substitute $c_1 = 3\sqrt{5}$ into the expression for c_3:

$$c_3 = 2(3\sqrt{5}) - 5\sqrt{5} = 6\sqrt{5} - 5\sqrt{5} = \sqrt{5}$$

Thus, the solution is:

$$c_1 = 3\sqrt{5}, \quad c_2 = 5, \quad c_3 = \sqrt{5}$$

Therefore, the vector $\mathbf{x} = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}$ can be expressed as:

$$\mathbf{x} = 3\sqrt{5}\hat{\mathbf{v}}_1 + 5\hat{\mathbf{v}}_2 + \sqrt{5}\hat{\mathbf{v}}_3$$