LAHORE UNIVERSITY OF MANAGEMENT SCIENCES
Department of Electrical Engineering

AI 501 Mathematics for Artificial Intelligence
Quiz 3 Solutions

Name: ______________________

Campus ID: ______________________

Total Marks: 10
Time Duration: 15 minutes

Question 1 (3 marks)

Given the vectors

$$v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 2 \\ 6 \end{pmatrix},$$

use the Gram-Schmidt process to find an **orthonormal** basis for the subspace spanned by these two vectors in $\mathbb{R}^2$.

Solution:

We are given two vectors:

$$v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$$

The goal is to apply the Gram-Schmidt process to find an orthogonal (or orthonormal) basis.

Step 1: First orthogonal vector u_1

The first orthogonal vector u_1 is simply the first vector v_1:

$$u_1 = v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Step 2: Second orthogonal vector u_2

To find the second orthogonal vector u_2, we need to subtract the projection of v_2 onto u_1 from v_2.

The projection of v_2 onto u_1 is given by:

$$\text{proj}_{u_1}(v_2) = \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1$$

First, compute the dot products:

$$v_2 \cdot u_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 6 \end{pmatrix} = 1 \times 2 + 3 \times 6 = 20$$

$$u_1 \cdot u_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 1^2 + 3^2 = 10$$

Now, calculate the projection:

$$\text{proj}_{u_1}(v_2) = \frac{20}{10} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$$

Now, subtract this projection from v_2 to find u_2:

$$u_2 = v_2 - \text{proj}_{u_1}(v_2) = \begin{pmatrix} 2 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Step 3: Normalize the vectors (optional)

To form an orthonormal basis, we can normalize u_1 and u_2.

1. **Normalize u_1:**

$$\|u_1\| = \sqrt{1^2 + 3^2} = \sqrt{10}$$

$$e_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{10}}\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{10}}\\\frac{3}{\sqrt{10}}\end{pmatrix}$$

2. **Normalize u_2:** No need to normalize as u_2 is the zero vector - this may be excluded from the final basis as it does not add any further information or contribute to the span of the set of vectors (as it has no direction).

Final Answer:

The orthonormal basis is:

$$\{e_1 = \begin{pmatrix}\frac{1}{\sqrt{10}}\\\frac{3}{\sqrt{10}}\end{pmatrix}\}$$

Question 2 (2 marks)

Determine if the following set forms a subspace of $\mathbb{R}^3$:

$$W = \{(x, y, z) \in \mathbb{R}^3 \mid x + y = z\}$$

Solution:

We are given the set $W = \{(x, y, z) \in \mathbb{R}^3 \mid x + y = z\}$. To determine whether W is a subspace of $\mathbb{R}^3$, we check three conditions:

1. **Contains the zero vector:** The zero vector in $\mathbb{R}^3$ is $(0, 0, 0)$. Substituting into the condition $x+y = z$:

$$0 + 0 = 0$$

Thus, the zero vector $(0, 0, 0) \in W$, so the first condition is satisfied.

2. **Closed under addition:** Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be two elements of W, which means:

$$x_1 + y_1 = z_1 \quad \text{and} \quad x_2 + y_2 = z_2$$

Now consider the sum:

$$(x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

We need to verify if:

$$(x_1 + x_2) + (y_1 + y_2) = z_1 + z_2$$

Using the conditions for W, we see:

$$(x_1 + y_1) + (x_2 + y_2) = z_1 + z_2$$

Therefore, W is closed under addition.

3. **Closed under scalar multiplication:** Let $(x, y, z) \in W$ and $c \in \mathbb{R}$. We need to check if $c(x, y, z) = (cx, cy, cz)$ is also in W. Substituting into the condition $x + y = z$:

$$cx + cy = c(x + y) = cz$$

Therefore, W is closed under scalar multiplication.

Since W contains the zero vector, is closed under addition, and is closed under scalar multiplication, we conclude that W is a subspace of $\mathbb{R}^3$.

Question 3 (5 marks)

Consider the block matrix

$$K = \begin{pmatrix} I & A^T \\ A & 0 \end{pmatrix}$$

where I is an identity matrix and 0 is a zero matrix. Which of the following statements **must** be true? For each statement, provide reasoning for your choice.

(a) K is square.

(b) A is square or wide.

(c) K is symmetric, i.e., $K^T = K$.

(d) The identity and zero submatrices in K have the same dimensions.

(e) The zero submatrix is square.

Solution:

Let $A \in \mathbb{R}^{m \times n}$.

It can be observed that $A^T \in \mathbb{R}^{n \times m}, I \in \mathbb{R}^{n \times n}, 0 \in \mathbb{R}^{m \times m}$ for the properties of block matrices to be preserved.

(a) K is square.

True. Adding up the size of K in both dimensions yields $m + n$, therefore $K \in \mathbb{R}^{(m+n) \times (m+n)}$

(b) A is square or wide.

False. A may also be a tall matrix without violating the properties of block matrices ($m > n$ does not lead to any contradictions).

(c) K is symmetric, i.e., $K^T = K$.

True.

$$K^T = \begin{pmatrix} I & A^T \\ A & 0 \end{pmatrix}^T$$

$$= \begin{pmatrix} I^T & A^T \\ (A^T)^T & 0^T \end{pmatrix}$$

$$= \begin{pmatrix} I & A^T \\ A & 0 \end{pmatrix} = K$$

(d) The identity and zero submatrices in K have the same dimensions.

False. The zero matrix in the bottom-right block of K may have different dimensions than the identity matrix I. If A is rectangular (not square), the zero matrix must accommodate the dimensions of A, which can differ from those of I. Thus, the identity and zero submatrices do not necessarily have the same dimensions.

(e) The zero submatrix is square.

True. As seen in (a), adding up the dimensions of I and A^T horizontally (number of columns), we have $n + m$, and adding the dimensions of I and A vertically (number of rows), we have $n + m$. Let 0 have dimensions $d_1 \times d_2$. We can add the number of columns in A and 0 to get $n + d_1 = n + m \to d_1 = m$. Similarly, adding the number of rows in A^T and 0 yields $n + d_2 = n + m \to d_2 = m$. As 0 has dimensions $m \times m$, it must be square.

Final Answers

- (a) True
- (b) False
- (c) True
- (d) False
- (e) True