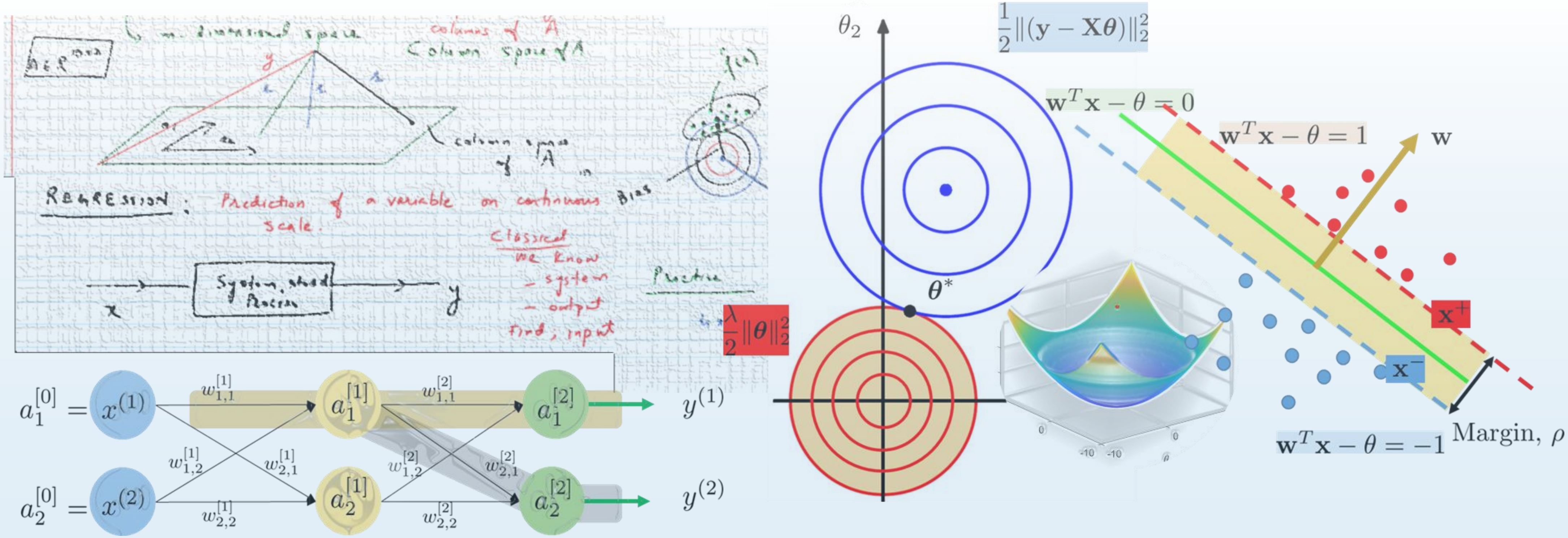

AI-501 Mathematics for AI

Eigenvalue Decomposition

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Outline

- Eigenvalue Decomposition
  - Eigenvectors, Eigenvalue overview
  - Formulation
  - Interpretation

Eigenvalue Decomposition (EVD)

Eigenvectors and Eigenvalues:

For square matrices, *eigenvectors* and *eigenvalues* are vectors and numbers represent the *eigen-decomposition* of a matrix; analyzes the structure of this matrix.

For a matrix $A \in \mathbf{R}^{n \times n}$

a vector $q \in \mathbf{R}^n, \ q \neq 0$, is called an eigenvector of A if

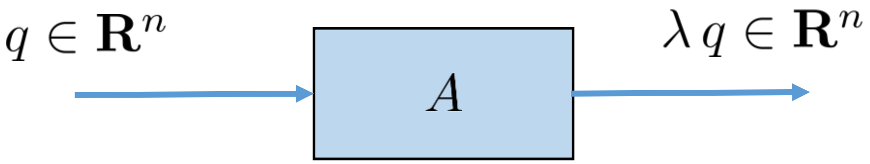
$$A\,q = \lambda\,q$$

Eigenvalue Equation

- λ is referred to as an eigenvalue of A associated with the eigenvector q

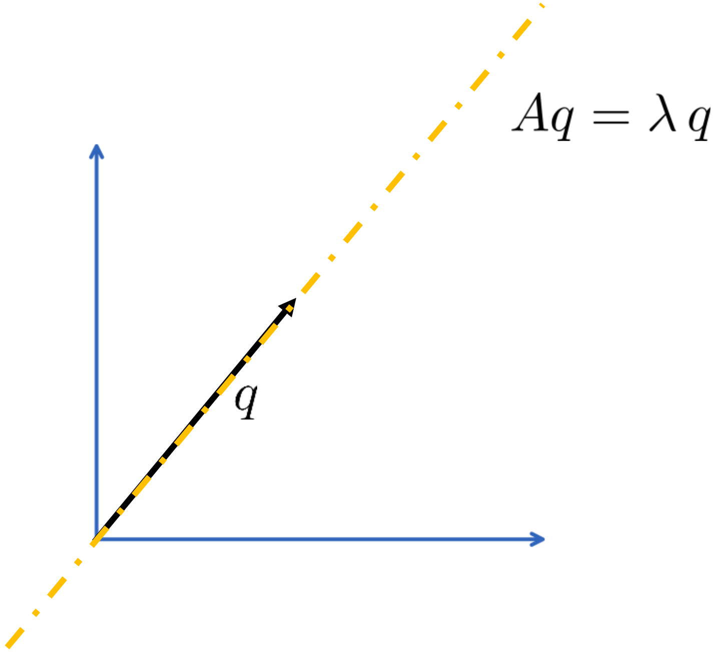
Eigenvalue Decomposition (EVD)

Linear transformation interpretation:



Only scaling

Graphically



Eigenvalue Decomposition (EVD)

Eigenvectors and Eigenvalues:

How to compute eigenvalues?

$$A\,q = \lambda\,q \qquad \Rightarrow \qquad Aq - \lambda q = 0$$

$$p(\lambda) = \det(Aq - \lambda q) = 0$$

Characteristic polynomial; degree *n*

n eigenvalues

Eigenspectrum: set of eigenvalues

n eigenvectors

Eigenspace: span of eigenevectors

Eigenvalue Decomposition (EVD)

Eigenvectors and Eigenvalues:

$$A\,q = \lambda\,q$$

n eigenvalues $\lambda_1,\ \lambda_2, \ldots, \lambda_n$

n eigenvectors $q_1,\ q_2, \ldots, q_n \in \mathbf{R}^n$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

$$A\,q_1 = \lambda_1\,q_1, \quad A\,q_2 = \lambda_2\,q_2, \ \ldots$$

$$Q = \begin{bmatrix} (q_1)_1 & (q_2)_1 & \ldots & (q_n)_1 \\ (q_1)_2 & (q_2)_2 & \ldots & (q_n)_2 \\ \vdots & \vdots & \ddots & \vdots \\ (q_1)_n & (q_2)_n & \ldots & (q_n)_n \end{bmatrix}$$

$$A\,Q = Q\,\Lambda$$

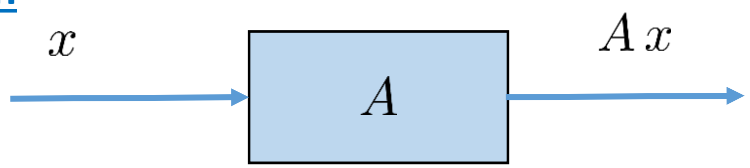
$$A = Q\,\Lambda\,Q^{-1}$$

Eigen-decomposition

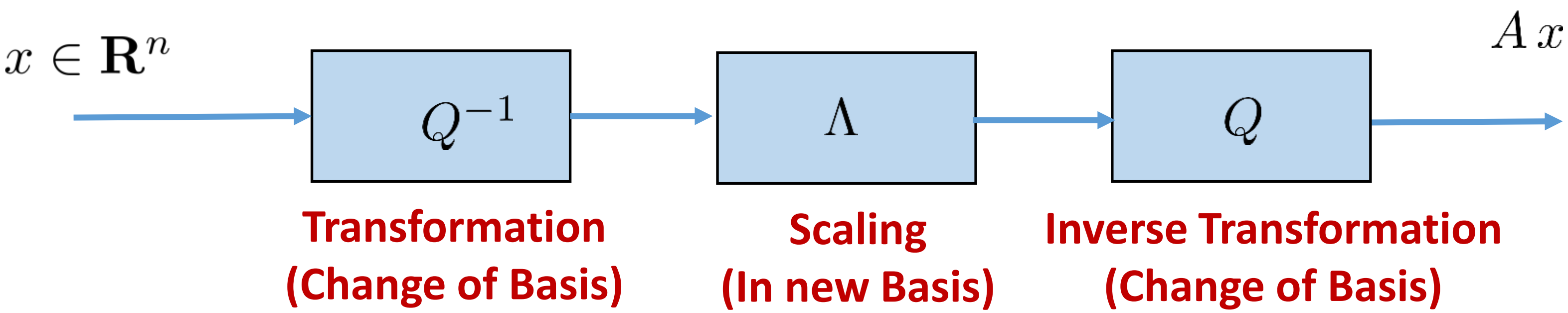
Eigenvalue Decomposition

What does eigenvector and eigenvalues reveal about A?

Linear Transformation

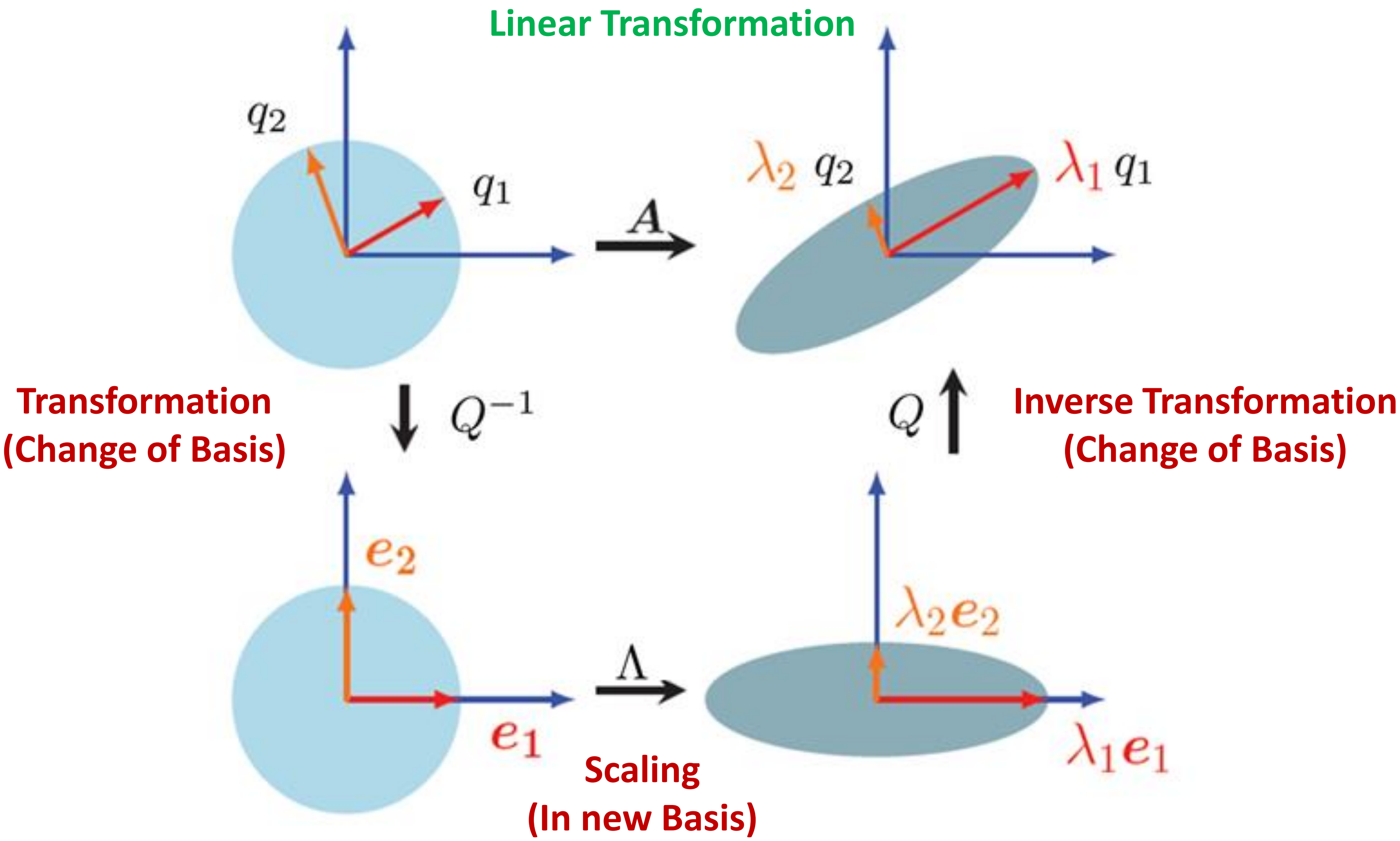


Linear Transformation Interpretation in terms of Eigen-decomposition of the Matrix



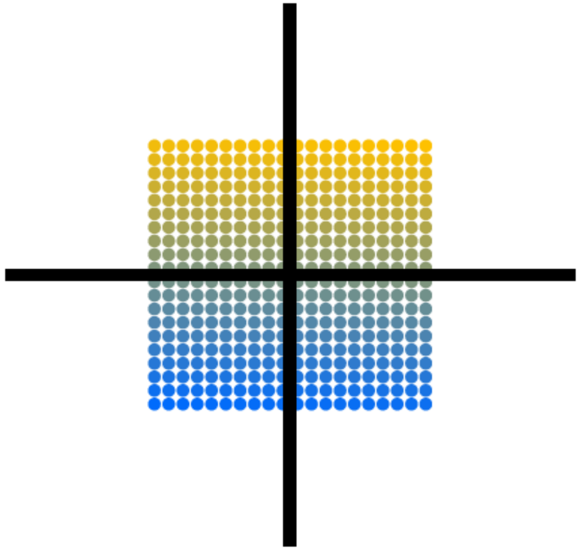
Eigenvalue Decomposition

Linear Transformation Interpretation in terms of Eigen-decomposition of the Matrix - Visualization

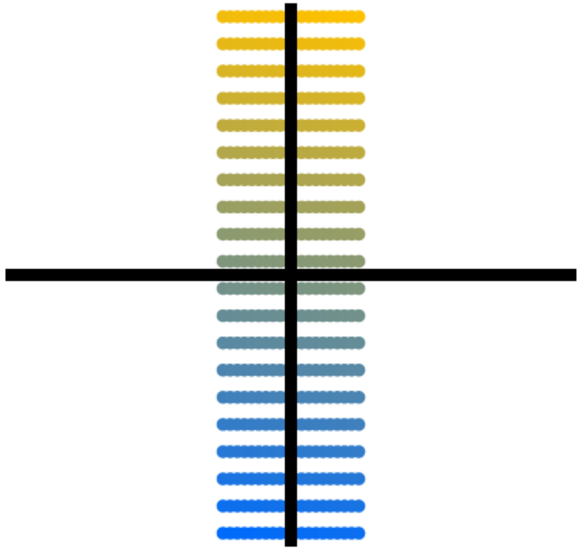


Eigenvalue Decomposition

Eigen-decomposition of the Matrix - Example

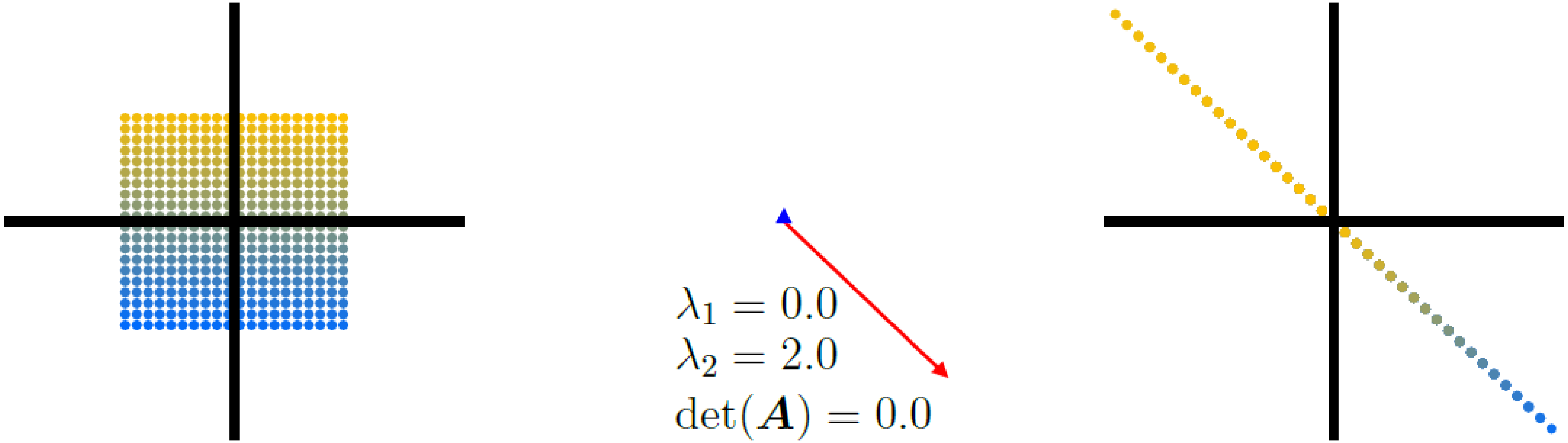


$$\lambda_1 = 2.0$$
$$\lambda_2 = 0.5$$
$$\det(\boldsymbol{A}) = 1.0$$



Eigenvalue Decomposition

Eigen-decomposition of the Matrix - Example



Eigenvalue Decomposition

Determinant in terms of Eigenvalues

- Determinant of a matrix $A \in \mathbf{R}^{n \times n}$ is given by the product of eigenvalues, that is,

$$\det(A) = \det(Q\Lambda Q^{-1}) = \det(Q)\det(\Lambda)\det(Q^{-1})$$

$$\det(A) = \det(Q\Lambda Q^{-1}) = \det(Q)\left(\prod_{i=1}^{n}\lambda_i\right)\frac{1}{\det(Q)}$$

$$\det(A) = \prod_{i=1}^{n}\lambda_i$$

- If one of the eigenvalues is zero, determinant of the matrix is zero.

  - I encourage you to connect this with the interpretation of determinant and (eigenvectors, eigenvalues).

Eigenvalue Decomposition (EVD)

EVD of Inverse Matrix:

- Matrix inverse A^{-1} has same eigenvectors but eigenvalues are given by inverse of the eigenvalues of the original matrix A.
- This can be shown in multiple ways. Let's use eigenvalue equation to show this.

For a matrix $A \in \mathbf{R}^{n \times n}$

$$A\,q = \lambda\,q$$

Eigenvalue Equation

- Assuming A is invertible, that is, all eigenvalues are non-zero.

$$A^{-1}A\,q = \lambda\,A^{-1}q \qquad \frac{1}{\lambda}q = A^{-1}q$$

This implies q is an eigenvector of A^{-1} with associated eigenvalue $\frac{1}{\lambda}$.

Eigenvalue Decomposition

Power of a matrix

$$AA = A^2 = Q\Lambda Q^{-1}\ Q\Lambda Q^{-1} = Q\Lambda^2 Q^{-1}$$

$$A^n = Q\Lambda^n Q^{-1}$$

- I encourage you to connect this with the interpretation of linear transformation using EVD of a matrix.

Eigenvalue Decomposition

Zero eigenvalues; Columns of A are not linearly independent

- If one of the eigenvalues is zero and q is an associated eigenvector.

$$Aq = \lambda q = 0$$

- It simply follows from the definition of linear independence that the columns are not linearly independent since Aq represents the linear combination of columns of A.

Eigenvalue Decomposition

Eigenvalues of a Symmetric Matrix

- (Spectral Theorem) For a symmetric matrix $A \in \mathbf{R}^{n\times n}$, there exists an orthonormal basis of the corresponding vectors space cossisting of eigenvectors of A and each eigenvalue is real.

$$A = Q\,\Lambda\,Q^{-1}$$

$$A^T = (Q^{-1})^T\,\Lambda\,Q^T$$

Since $A = A^T$, we have $(Q^{-1} = Q^T$

⇩

Orthonormal matrix

- Summary: For a symmetric matrix, eigenvectors are orthonromal and eigenvalues are real.