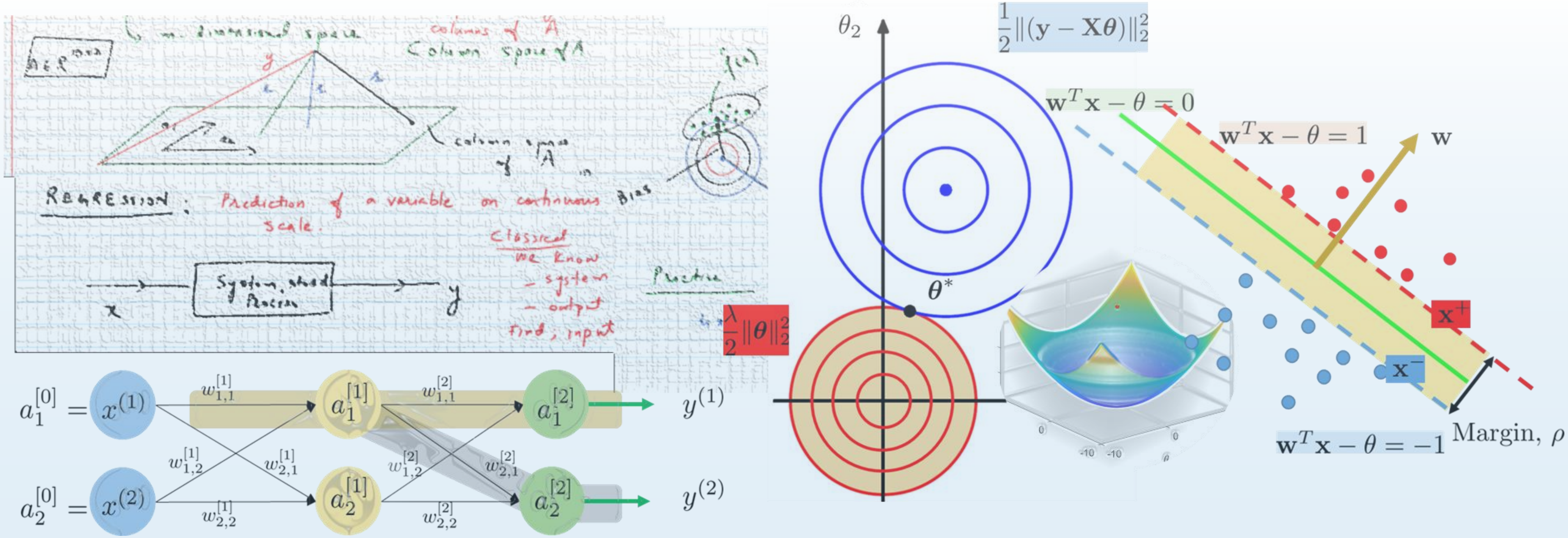

AI-501 Mathematics for AI

Convex Optimization – Overview

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Convex Optimization – Overview

Optimization – Overview:

Optimization involves finding the best solution to a problem within a set of constraints. Formally, an optimization problem can be expressed as:

$$\min_{x \in \mathbb{R}^n} f_o(x) \quad \text{subject to} \quad x \in \mathcal{C},$$

where:

- $f_o(x)$ is the **objective function**, which we aim to minimize or maximize.
- $\mathcal{C}$ is the **feasible set**, defined by constraints such as inequalities or equalities.
- x represents the **decision variables**.

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Optimization – Overview – Formulation:

An optimization problem of finding a variable x is usually formulated as

$$\begin{aligned} \text{minimize} \quad & f_o(x) \\ \text{subject to} \; & f_i(x) \leq 0, \quad i = 1, 2, \ldots, m \\ & h_j(x) = 0, \quad , j = 1, 2, \ldots, p \end{aligned}$$

$f_o(x)$ - Objective function $\quad f_i(x)$ - Inequality constraint functions $\quad h_j(x)$ - equality constraint functions

Applications:

Machine Learning: Training models, support vector machines, and regression.

Engineering: Signal processing, control systems, and circuit design.

Economics: Portfolio optimization and game theory.

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Optimization – Overview – Examples:

Unconstrained Optimization:

$$\min_{x \in \mathbb{R}} (x^2 - 4x + 4).$$

Constrained Optimization:

$$\min_{x_1, x_2 \in \mathbb{R}^2} x_1^2 + x_2^2 \quad \text{subject to} \quad x_1 + x_2 = 1,\ x_1 \geq 0,\ x_2 \geq 0.$$

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Standard Form of Convex Optimization Problems:

The general form of a convex optimization problem is:

$$\begin{aligned} &\text{minimize} \quad f_o(x) \\ &\text{subject to} \; f_i(x) \leq 0, \quad i = 1, 2, \ldots, m \\ &\qquad\qquad\;\; h_j(x) = 0, \quad , j = 1, 2, \ldots, p \end{aligned}$$

where:

$f_o(x)$ is a convex objective function.

$f_i(x)$ are convex inequality constraint functions.

$h_j(x)$ are affine equality constraint functions.

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Convex Optimization Problems – Geometric Interpretation:

Convex optimization can be viewed as finding the "lowest point" of the graph of a convex objective function $f_o(x)$ over a feasible region $\mathcal{C}$. The feasible region is defined as the intersection of the constraints:

$$\mathcal{C} = \{x \in \mathbb{R}^n \mid f_i(x) \leq 0,\ h_j(x) = 0\}.$$

Connection to Convex Sets:

If all $f_i(x)$ are convex functions and all $h_j(x)$ are affine functions, the feasible region $\mathcal{C}$ is a convex set. This is because:

- A convex inequality constraint $f_i(x) \leq 0$ defines a convex region.
- Affine equality constraints $h_j(x) = 0$ define a linear subspace or affine subspace, which is convex.
- The intersection of convex sets is convex.

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Convex Sets – Overview here – See Tutorial Problem Notes:

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Convex Optimization Problems – Geometric Interpretation:

Visualization: In $\mathbb{R}^2$ (two dimensions), the convex optimization problem can be visualized as:

A convex feasible region $\mathcal{C}$, often represented as a polygon or curved region.

A convex objective function $f(x)$, represented as contour lines (level sets) or a surface.

The goal is to find the point within $\mathcal{C}$ where $f(x)$ achieves its minimum value.

Optimal Solution: If the feasible region $\mathcal{C}$ is non-empty and $f(x)$ is continuous and convex, the optimization problem has a global minimum at a point $x^* \in \mathcal{C}$. This property arises from the convexity of $f(x)$ and $\mathcal{C}$, which ensures no local minima exist outside the global minimum.

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Convex Optimization Problems – Geometric Interpretation – Examples:

Linear Programming (LP): The feasible region $\mathcal{C}$ is a polyhedron, and the objective function $f(x) = c^T x$ is linear. The optimal solution is at a vertex of the polyhedron.

Quadratic Programming (QP): The feasible region $\mathcal{C}$ is convex, and the objective function $f(x) = \frac{1}{2}x^T Qx + c^T x$ is a convex paraboloid. The optimal solution lies in the interior or on the boundary of $\mathcal{C}$.

Let's cover this in more detail.

Convex Optimization – Overview

Linear Programming (LP):

Linear programming is a special case of convex optimization where both the objective function and the constraints are linear. The standard form with inequality and equality constraints is:

$$\min_{x \in \mathbb{R}^n} \; c^T x$$

$$\text{subject to:} \quad Gx \leq h, \quad Ax = b.$$

$c \in \mathbb{R}^n$ is the coefficient vector of the objective function.

$G \in \mathbb{R}^{m_{\text{ineq}} \times n}$ is the constraint matrix for the inequality constraints.

$h \in \mathbb{R}^{m_{\text{ineq}}}$ is the vector of bounds for the inequality constraints.

$A \in \mathbb{R}^{m_{\text{eq}} \times n}$ is the constraint matrix for the equality constraints.

$b \in \mathbb{R}^{m_{\text{eq}}}$ is the vector of bounds for the equality constraints.

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Linear Programming (LP) – Example:

Problem: Minimize the cost of production subject to resource constraints.

$$\min_{x_1, x_2} \; 3x_1 + 5x_2 \quad \text{subject to:}$$

$$2x_1 + x_2 \leq 6, \quad x_1 + 3x_2 \leq 9, \quad x_1, x_2 \geq 0.$$

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Quadratic Programming (QP):

Quadratic programming extends linear programming by allowing the objective function to be quadratic, while constraints remain linear. The standard form is:

$$\text{minimize } f_o(x) = \frac{1}{2}x^T P x + q^T x + r \quad \text{Quadratic} \quad \text{(Convex)} \quad P \in \mathbf{S}_+^n$$

$$\left.\begin{aligned} \text{subject to } & Ax \preceq b \\ & Gx = h \end{aligned}\right\} \text{Affine}$$

Convex Optimization – Overview

Quadratic Programming (QP) – Examples:

1. Least-Squares:

$$\text{minimize } f_o(x) = \|Ax - b\|_2^2$$

Quadratic

2. Constrained Least-Squares:

$$\text{minimize } f_o(x) = \|Ax - b\|_2^2$$

Quadratic

$$\text{subject to } a \preceq x \preceq b$$

Box Constraints