LAHORE UNIVERSITY OF MANAGEMENT SCIENCES
Department of Electrical Engineering

EE212 Mathematical Foundations of Machine Learning and Data Science
Quiz 04 Solution

Total Marks: 15 **Time Duration:** 45 minutes

Question 1 (3 marks)

The following set of equations can be modeled as $Ax = b$.

$$\begin{aligned} 3x + 5y &= 1 \\ x - 2y &= 3 \\ 2x + 5y - z &= 7 \\ 3y - 4z &= 2 \end{aligned}$$

(a) Write down the matrix A.

(b) Give an *expression* for x in terms of A and b.

Solution:

(a)

$$A = \begin{bmatrix} 3 & 5 & 0 \\ 1 & -2 & 0 \\ 2 & 5 & -1 \\ 0 & 3 & -4 \end{bmatrix}$$

(b) $x = A^{\dagger} b$
$x = (A^T A)^{-1} A^T b$

Question 2 (5 marks)

Consider the following matrix:

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

(a) Compute the gram matrix.

(b) Determine if the gram matrix is invertible or not.
Hint: You don't need to compute the determinant of the 3×3 matrix.

(c) What role do the eigen-values and eigen-vectors of gram matrix play in the SVD of A?

Solution:

(a)

$$G = \begin{bmatrix} 5 & 1 & 4 \\ 1 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix}$$

(b) Gram matrix for a wide matrix will always be non-invertible as it will have at least one zero eigen-value.

(c) eigen-values(G) are squared singular values in Σ
eigen-vectors(G) form right singular vectors

Question 3 (4 marks)

A 3 by 3 matrix B is known to have eigenvalues 0, 1, 2. Is this information enough to determine the following? Give answers where possible:

(a) the determinant of $B^T B$

(b) the eigenvalues of $(B^2 + I)^{-1}$

Solution:

(a) We have $\det B^T B = \det B^T \det B = (\det \mathrm{B})^2 = 0 \cdot 1 \cdot 2 = 0$.

(b) If $Ax = \lambda x$, then $x = \lambda A^{-1} x$; also, any polynomial $p(t)$ yields $p(A)x = p(\lambda)x$. Hence the eigenvalues of $(B^2 + I)^{-1}$ are $1/(0^2 + 1)$ and $1/(1^2 + 1)$ and $1/(2^2 + 1)$, or 1 and 1/2 and 1/5.

Question 4 (3 marks)

A is an $m \times n$ matrix of rank r. Suppose there are right sides b for which $Ax = b$ has *no solution*.

(a) What are all the inequalities ($<$ or $\leq$) that must be true between m, n and r?

(b) How do you know that $A^T y = 0$ has solutions other than $y = 0$?

Solution:

(a) The rank of a matrix is always less than or equal to the number of rows and columns, so $r \leq m$, and $r \leq n$. Moreover, by the second statement, the column space is smaller than the space of possible output matrices, i.e. $r < m$.

(b) These solutions make up the left nullspace, which has dimension $m - r > 0$ (that is, there are nonzero vectors in it).