LAHORE UNIVERSITY OF MANAGEMENT SCIENCES
Department of Electrical Engineering

EE212 Mathematical Foundations of Machine Learning and Data Science
Quiz 05 Solution

Total Marks: 10 **Time Duration:** 45 minutes

Question 1 (3 marks)

A has 4 eigen values, 2 of them are non-zero.

(a) What is the dimension of null space of A?

(b) Using EVD, show that the dimension of the null space of A^2 is the same as that of null space of A.

Solution:

(a) A has 2 eigen values which are 0, do dimension of null space is 2.

(b)

$$A = S\Lambda S^{-1}$$

$$A^2 = (S\Lambda S^{-1})(S\Lambda S^{-1}) = S\Lambda^2 S^{-1}$$

Hence, A^2 has the same number of non-zero eigen values and the same corresponding eigen vectors, thus the dimension of null space of A^2 is also 2.

Question 2 (2 marks)

Define rank of a matrix. How do we determine the rank of a matrix using its singular value decomposition?

Solution: Rank of a matrix is given by the number of linearly independent rows that is equal to the number of linearly independent columns. Rank of a matrix is equal to the number of non-zero singular values of the matrix.

Question 3 (5 marks)

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

The above matrix can be represented as $A = U\Sigma V^T$, where U, Σ and V are all 2 x 2 matrices. This matrix has singular values $\sigma_1 = 2\sqrt{2}$ and $\sigma_2 = \sqrt{2}$ and corresponding to the first singular value, the right singular vector is:

$$v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

(a) Find v_2 which is the second normalized right singular vector.

(b) Find U.

Solution:

(a) v_2 is orthogonal to v_1 so let $v_2 = [a, b]^T$. We know $v_1^T v_2 = 0$ due to orthogonality. So $a + b = 0$ thus $a = -b$. We also know that v_2 is normalized, so $a^2 + b^2 = 1$. Solving these equations simultaneously we get:

$$v_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

(b) $u_1 = \frac{Av_1}{\sigma_1}$ and $u_2 = \frac{Av_2}{\sigma_2}$. From these equations we get:

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$