LUMS
A Not-for-Profit University

Department of Electrical Engineering
School of Science and Engineering

EE212 Mathematical Foundations for Machine Learning and Data Science

ASSIGNMENT 3 - Solution

Due Date: 23:55, Monday, November 22, 2021. (Submit online on LMS)
Format: 8 problems, for a total of 100 marks
Instructions:

- You are allowed to collaborate with your peers but copying your colleague's solution is strictly prohibited. This is not a a group assignment. Each student must submit his/her own assignment, scanned in **a single PDF document.**
- Solve the assignment on blank A4 sheets and either scan the document using a scanner or use CamScanner proficiently.
- Upload the solved assignment on LMS in the "Assignments" tab under Assignment 01.
- Naming convention should be as follows: "Name_RollNumber_Assignment_1.pdf"
- Feel free to contact the instructor or the teaching assistants if you have any concerns.

- You represent the most competent individuals in the country, do not let plagiarism come in between your learning. In case any instance of plagiarism is detected, the disciplinary case will be dealt with according to the university's rules and regulations.

Problem 1 (12 marks)

(a) [**1 mark**] Given $y = \begin{bmatrix} 0 & 1 & 4 & 9 & 16 & 25 & 36 & 49 \end{bmatrix}$, plot y.

(b) [**3 marks**] Taking a look at the graph of part (a), what do you expect the rate of change (or the slope) to be at every point?

(c) [**3 marks**] Find the values of y' which is the first difference of y.

(d) [**2 marks**] Plot y'.

(e) [**3 marks**] Comment on the relationship between the plots from part (a) and part (d).

Solution.

(a) Against each index on the x-axis (i.e. 1,2,3), plot the corresponding y-value from the data given. The coordinates will be (1,0), (2,1), (3,4), and so on. Draw vertical lines to denote the magnitudes against each index value touching the coordinates that form.

(b) At every point, the slope can be observed to increase. In other words, the corresponding rate of change increases as the index increases.

(c) $y' = \begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 & 13 \end{bmatrix}$

(d) Against each index on the x-axis (i.e. 1,2,3,..), plot the corresponding y-value from the new data set from part (c). The coordinates will be (1,1), (2,3), (3,5), and so on. Draw vertical lines to denote the magnitudes against each index value touching the coordinates that form.

(e) y' follows a linear increase as expected. y is an x^2 graph and y should be something similar to its derivative (i.e. $2x$). A linear dependence is observed in the first difference as expected.

Problem 2 (13 marks)

(a) [**1 mark**] Given $z = \begin{bmatrix} 1 & 15 & 7 & 1 & 21 & 9 & 5 & 10 & 15 & 8 \end{bmatrix}$, plot z.

(b) [**4 marks**] Find the 2 point moving average and plot the resultant data.

(c) [**4 marks**] Find the 5 point moving average and plot the resultant data.

(d) [**4 marks**] Discuss the two plots and their differences in terms of "details" and "trends" of data.

Solution.

(a) Against each index on the x-axis (i.e. 1,2,3), plot the corresponding z-value from the data given. The coordinates will be (1,1), (2,15), (3,7), and so on. Draw vertical lines to denote the magnitudes against each index value touching the coordinates that form.

(b) $z_2 = \begin{bmatrix} 0.5 & 8 & 11 & 4 & 11 & 15 & 7 & 7.5 & 12.5 & 11.5 & 4 \end{bmatrix}$.
Plot a graph of these values against each index as done in part (a).

(c) $z_5 = \begin{bmatrix} 0.2 & 3.2 & 17.4 & 4.8 & 9 & 10.6 & 8.6 & 9.2 & 12 & 9.4 & 7.6 & 6.6 & 4.6 & 1.6 \end{bmatrix}$.
Plot a graph of these values against each index as done in part (a).

(d) Higher point moving averages smooth out the data more. The "details" and sharp changes in the data are lost but a smooth and clear general "trend" of the data can be observed. Too high moving point averages can be detrimental as useful information can be lost. Similarly, too low moving point averages can be useless as they may carry too much noise.

Problem 3 (15 marks)

Consider two coins: Coin A and Coin B. Coin A is a fair coin (contains heads and tails with equal probability of landing on each). Coin B is a two-headed coin (contains heads on both sides). You are asked to randomly pick one of the two coins and toss it. What is the probability that if the top side of the coin lands on heads that the other side (facing downwards) is also heads?

Solution.
Define two events as follows.
A: Event that the two-headed coin is picked and tossed.
B: Event that the top side shows heads.
This question asks to find $P(A|B)$.
Using Baye's Rule, $P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{1\times0.5}{(1\times0.5)+(0.5\times0.5)} = \frac{2}{3}$.

Problem 4 (15 marks)

Hoping to win the semi finals so that they can make it to the finals of the ICC T20 World Cup, the Pakistani team decides to choose a line-up of 11 cricket players for their semi-finals match out of the total of 15 players that have travelled to the UAE for the tournament. In how many ways can 11 Pakistani cricket players be chosen if:

(a) [**5 marks**] There is no restriction on the selection.

(b) [**5 marks**] Asif Ali is always chosen.

(c) [**5 marks**] Asif Ali is never chosen.

Solution.

(a)

$$^{15}C_{11}$$

(b)

$$^{14}C_{10}$$

(c)

$$^{14}C_{11}$$

Problem 5 (15 marks)

A biased die is thrown 30 times and of these 30 throws, 8 result in the number 6. Suppose that this die is rolled another 12 times.

(a) [**5 marks**] Find the probability that a 6 will be rolled exactly twice in the additional 12 throws.

(b) [**5 marks**] Find the expected number of sixes.

(c) [**5 marks**] Find the variance of the number of sixes.

Solution.

(a) Let S be defined by $S \sim B(12, p)$, outlining the number of sixes observed in 12 throws where $p = \frac{8}{30} = \frac{4}{15}$.
Since $S \sim B(12, \frac{4}{15})$,
$P(S = 2) = \binom{12}{2} (\frac{4}{15})^2(\frac{11}{15})^{10} = \frac{66\times4^2\times11^{10}}{15^{12}} = 0.211$

(b) $E(S) = np = 12 \times \frac{4}{15} = 3.2$

(c) $V(S) = npq = 12 \times \frac{4}{15} \times \frac{11}{15} = 2.347$

Problem 6 (10 marks)

A computer program produces symbols at random from a set $S = a, b, c, d, e$ with the following probabilities: $P(a) = \frac{1}{2}$, $P(b) = \frac{1}{4}$, $P(c) = \frac{1}{8}$, $P(d) = P(e) = \frac{1}{16}$. A data compression algorithm then encodes the symbols into binary strings as follows:

$$\begin{aligned} a &= 1 \\ b &= 01 \\ c &= 001 \\ d &= 0001 \\ e &= 0000 \end{aligned}$$

Let the random variable Y be equal to the length of the binary string output of the system (i.e. the length of "1" is 1 and the length of "01" is 2, and so on). Find the Probability Mass Function (PMF) of Y.

Solution.

$y =\in \{1, 2, 3, 4\}$
$P_Y(1) = P(a) = \frac{1}{2}$
$P_Y(2) = P(b) = \frac{1}{4}$
$P_Y(3) = P(c) = \frac{1}{8}$
$P_Y(4) = P(d) + P(e) = \frac{1}{16} + \frac{1}{16} = \frac{1}{8}$

$$P_Y(y) = \begin{cases} \frac{1}{2} & \text{; for y=1} \\ \frac{1}{4} & \text{; for y=2} \\ \frac{1}{8} & \text{; for y=3 or y=4} \end{cases} \tag{1}$$

Problem 7 (10 marks)

Given a random variable R such that $R \sim B(1 - r^4)$ for $-1 \leq r \leq 1$:

(a) **[3 marks]** Find B.

(b) **[3 marks]** Find the CDF pf R.

(c) **[4 marks]** Find $P\{|x| \leq \frac{1}{2}\}$

Solution.

(a) (Normalization)

$$\int_0^\infty f(x)\,dx = B\int_{-1}^{1}(1 - x^4)\,dx = 1$$

$$B = \frac{5}{8}$$

(b) (Use a dummy variable i.e. t)

$$F_X(x) = \frac{5}{8}\int_{-1}^{x}(1 - t^4)\,dt = \frac{5}{8}(x - \frac{x^5}{5} + \frac{4}{5}) \text{ for } -1 \leq x \leq 1$$

(c)

$$P|x| < \frac{1}{2} = P\frac{-1}{2} \leq x \leq \frac{1}{2} = \frac{5}{8}\int_{\frac{-1}{2}}^{\frac{1}{2}}(1 - x^4)\,dx = \frac{79}{128}$$

Problem 8 (10 marks)

Let D be a discrete random variable with the following PMF:

$$P_D(k) = \begin{cases} 0.1 & \text{for k=0} \\ 0.4 & \text{for k=1} \\ 0.3 & \text{for k=2} \\ 0.2 & \text{for k=3} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

(a) **[3 marks]** Find the expected value of D.

(b) **[3 marks]** Find $Var(D)$.

(c) **[4 marks]** If $A = (D-2)^2$, find the expected value of A.

Solution.

(a) $EX = \sum_{xk \,\epsilon\, R_x} x_k P_x(x_k) = 0(0.1) + 1(0.4) + 2(0.3) + 3(0.2) = 1.6$

(b) $Var(X) = EX^2 - (EX)^2 = EX^2 - (1.6)^2$ can be used.
$EX^2 = 0^2(0.1) + 1^2(0.4) + 2^2(0.3) + 3^2(0.2) = 3.4$.
Thus, $Var(X) = (3.4) - (1.6)^2 = 0.84$

(c) $E(X-2)^2 = (0-2)^2(0.1) + (1-2)^2(0.4) + (2-2)^2(0.3) + (3-2)^2(0.2) = 1$.

— End of Assignment —