LUMS
A Not-for-Profit University

Department of Electrical Engineering
School of Science and Engineering

EE212 Mathematical Foundations for Machine Learning and Data Science

ASSIGNMENT 1

Due Date: 13:30, Tuesday, September 27, 2022.
Format: 12 problems, for a total of 100 marks
Instructions:

- You are allowed to collaborate with your peers but copying your colleague's solution is strictly prohibited. This is not a group assignment. Each student must submit his/her own assignment.
- Solve the assignment on blank A4 sheets and staple them before submitting.
- Submit in-class or in the dropbox labeled EE-212 outside the instructor's office.
- Write your name and roll no. on the first page.
- Feel free to contact the instructor or the teaching assistants if you have any concerns.

- You represent the most competent individuals in the country, do not let plagiarism come in between your learning. In case any instance of plagiarism is detected, the disciplinary case will be dealt with according to the university's rules and regulations.

Problem 1 (8 marks)

Given the vectors below,

$$a = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Calculate the following.

(a) [**2 marks**] $2a + b$

(b) [**2 marks**] $a + b - c$

(c) [**2 marks**] $(a + b) \cdot c$, where '$\cdot$' denotes the dot product.

(d) **[2 marks]** Let's say you ask your friend how to do parts b and c, and they suggest that you extend the vector c with zero, e.g.,:

$$c = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Explain why this is not a good approach.

Problem 2 (12 marks)

Matrix vector product ($Ax = b$) represents just a system of linear equations.

Express each of the following as an explicit system of linear equations, and solve them.

If an exact solution does not exist, comment on why it does not exist and find the best possible solution.

(a) **[2 marks]** $\begin{bmatrix} x \\ y \end{bmatrix} - 3 \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(b) **[2 marks]** $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(c) **[2 marks]** $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(d) **[3 marks]** $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(e) **[3 marks]** $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 1 \\ 8 \end{bmatrix}$

Problem 3 (10 marks)

An inner product of two vectors is defined as follows:

$$< u, v >= u^T v = \sum_{i=1}^{n} u_i v_i$$

For it to be a valid inner product it must satisfy certain conditions,

1. Linear: $< a\vec{u} + b\vec{v}, \vec{w} >= a < \vec{u}, \vec{w} > +b < \vec{v}, \vec{w} >$, where a and b are any scalars.
2. Symmetric: $< \vec{u}, \vec{v} >=< \vec{v}, \vec{u} >$
3. Positive Definite: $< \vec{u}, \vec{u} > \geq 0$ and $< \vec{u}, \vec{u} >= 0$ iff $\vec{u} = 0$
   Or simply, vectors always have a non-negative magnitude.

Suppose that we define a weighted inner product on $\mathbf{R}^2$ as follows: $\langle x, y \rangle = 4x_1y_1 + 5x_2y_2$.
Note that this is different from the standard inner product on $\mathbf{R}^2$.
Show that the inner product we defined is valid i.e., it satisfies the 3 conditions.

Problem 4 (10 marks)
Given a vector in $\mathbf{R}^2$.

$$\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Find all vectors that form an $\angle 45$ to $\vec{a}$. Show your working.

Problem 5 (6 marks)
Given,

$$A \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

(a) **[3 marks]** Find the matrix A.

(b) **[3 marks]** Can you find A without calculating the inverse of X, where $X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

Problem 6 (4 marks)
Understanding sparsity.

(a) **[2 marks]** Give example of a 3×3 matrix that is full rank, but also maximally sparse.

(b) **[2 marks]** Give examples of data collected from daily life that will result in a sparse data matrix.

Problem 7 (8 marks)
Given the vectors in $\mathbf{R}^3$

$$a = \begin{bmatrix} 7 \\ -3 \\ 4 \end{bmatrix}, b = \begin{bmatrix} 0 \\ -9 \\ 1 \end{bmatrix}, c = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

(a) **[3 marks]** Determine the vector $\vec{c}$ such that the set $\{a, b, c\}$ is linearly dependent.

(b) **[3 marks]** Determine the vector $\vec{c}$ such that the set $\{a, b, c\}$ is linearly independent.

(c) **[2 marks]** Find another vector such that the set $\{a, b, c, d\}$ is linearly independent.

Problem 8 (10 marks)
You are on the coordination committee of O-Week and are looking for coaches. There are five qualities that you think should be in a coach or at least four of them. Enthusiastic, energetic, good sense of humor, friendly, and gives good guidance. You receive the following four applications,

1. Abdullah who is energetic and enthusiastic, gives good guidance but is not friendly and has a poor sense of humor.
2. Aamnah who is super friendly, gives great advice, is energetic and enthusiastic but lacks a good sense of humor.
3. Ibrahim who has a good sense of humor and easily makes friends, is also energetic and enthusiastic, but is not good at giving advises.
4. Finally Ayesha who was a coach last year as well and excels in all five departments.

(a) [**3 marks**] Represent the data in a data matrix as such,
1 if they have the quality and 0 if they don't.

(b) [**4 marks**] Automate the selection process, i.e., represent it as a matrix-vector product.
hint: The matrix-vector product gives you the number of qualities in each applicant. Then you can compare each entry with a threshold which gives you a binary vector, with a 1 if you should select the coach and a 0 if you should not.

(c) [**3 marks**] The implementation you did in part (a) (if done correctly) suggests you 3 applicants to select, but you only need 2. Change the selection criteria a little. Giving different importance to different qualities.
hint: Being able to give good guidance is more important than having a good sense of humor.

Problem 9 (12 marks)

Consider the following matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}.$$

(a) [**4 marks**] Find four fundamental subspaces of A. Show your working.

(b) [**4 marks**] Given that A is an $n \times m$ matrix and is full-rank, what are the dimensions of all four subspaces of A in terms of n, m, and R. Where R is the rank of A.

(c) [**4 marks**] Take a vector in $\mathbf{R}^2$, that has some component in both the column space of A^T and the null space of A, e.g.,

$$\begin{bmatrix} -2 \\ 9 \end{bmatrix}$$

Apply the transformation A on it. Now apply the transformation A on its two components from the column space of A^T and null space of A and provide an interpretation of the results.

Problem 10 (8 marks)

Gram-Schmidt orthogonalization.
Given a hyperplane in $\mathbf{R}^3$, spanned by

$$\vec{a} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

(a) [**3 marks**] Find its orthonormal basis.

(b) [**5 marks**] Extend the set of basis found in part a, such that it is an orthonormal basis for $\mathbf{R}^3$.

Problem 11 (4 marks)

Given an equilateral triangle in $\mathbf{R}^2$ with the following vertices,

$$A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$$

(a) [**2 marks**] Apply a linear transformation on the vertices using the matrix $X = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.
Do the transformed vertices still correspond to that of an equilateral triangle? What is the relation between the area of the triangle before and after the linear transformation.

(b) **[2 marks]** Now, another linear transformation by the matrix $Y = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

What is the relation between the area of the original triangle and the triangle after the linear transformations X and Y.

Problem 12 (8 marks)

There are 3 types of tickets being sold at a concert; standard, VIP and a golden ticket that guarantees a meet and greet with the artist.

You are given the number of each type of ticket sold on the first two days of the ticket sale, and the total amount collected on both days.

Day 1: 500 standard, 50 VIP, and 4 golden.

Day 2: 700 standard, 20 VIP, and 1 golden.

Total amount collected on the two days was 320,000 PKR and 355,000 PKR.

(a) What is the price of each of the 3 categories of tickets?

(b) Now you ask the management for the actual prices and they are,

$$\text{standard} = 500 \text{ PKR, VIP} = 1000 \text{ PKR, Golden} = 5000 \text{ PKR}$$

Were you able to find the actual prices in part a?

— End of Assignment —