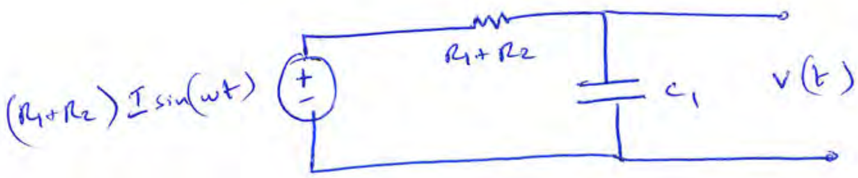
EE 240 — Circuits 1
Assignment 5 solution

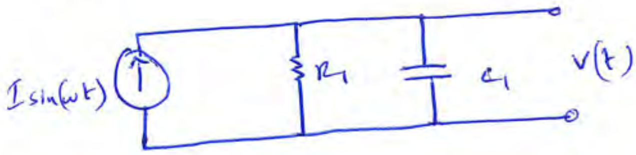
Ans 1: at $t < 0$



when capacitor charged → oc

$$\boxed{V(0^-) = I\sin(\omega t)\,(R_1+R_2)} = \boxed{0}$$

for $t > 0$



KCL

$$C_1 \frac{dv(t)}{dt} + \frac{v(t)}{R_1} = I\sin(\omega t)$$

$$\frac{dv(t)}{dt} + \frac{[v(t)]}{R_1C_1} = \frac{I\sin(\omega t)}{C_1}$$ non-homogeneous

complementary solution:

$$\frac{dv(t)}{dt} = -\frac{1}{R_1C_1}v(t)$$

$$\boxed{v(t) = K e^{-t/R_1C_1}}$$

particular

$$v_p(t) = K_1\sin(\omega t) + K_2\cos(\omega t) \quad -(1)$$

substitute (1) in KCL:

$$\omega k_1 \cos(\omega t) - \omega k_2 \sin(\omega t) + \frac{1}{R_1 C_1}\left[k_1 \sin(\omega t) + k_2 \cos(\omega t)\right] = \frac{I \sin(\omega t)}{C}$$

equate coefficients:

$\cos(\omega t)$ | $\omega k_1 + \frac{k_2}{R_1 C_1} = 0$

$$\boxed{k_1 = \frac{-k_2}{\omega R_1 C_1}} \quad - (2)$$

$\sin(\omega t)$ | $-\omega k_2 + \frac{k_1}{R_1 C_1} = \frac{I}{C_1}$

$$\boxed{k_2 = \frac{k_1}{\omega R_1 C_1} - \frac{I}{\omega C_1}} \quad - (3)$$

substitute (3) in (2):

$$k_1 = \frac{-1}{\omega R_1 C_1}\left[\frac{k_1}{\omega R_1 C_1} - \frac{I}{\omega C_1}\right]$$

$$\boxed{k_1 = \frac{I R_1}{\omega^2 R_1^2 C_1^2 + 1}} \quad - (4)$$

substitute (4) in (3)

$$\boxed{k_2 = -\frac{\omega R_1^2 C_1 I}{\omega^2 R_1^2 C_1^2 + 1}}$$

$$\boxed{v(t) = k e^{-\frac{t}{R_1 C_1}} + \left(\frac{I R_1}{\omega^2 R_1^2 C_1^2 + 1}\right)\sin(\omega t) - \left(\frac{\omega R_1^2 C_1 I}{\omega^2 R_1^2 C_1^2 + 1}\right)\cos(\omega t)}$$

Find k:

at $t = 0^+$ $\quad v(0^+) = I \sin(\omega t)\,[R_1 + R_2]$

$$k - \frac{\omega R_1^2 C_1 I}{\omega^2 R_1^2 C_1^2 + 1} = I \sin(\omega t)\,[R_1 + R_2] = 0$$

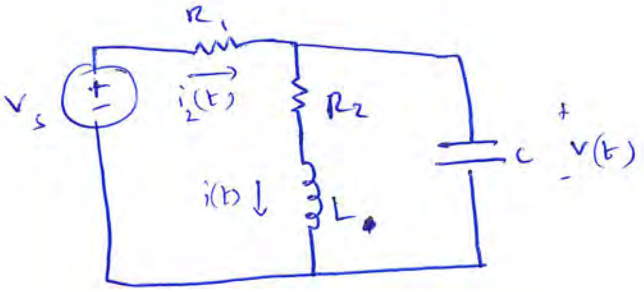
$$k = I \sin(\omega 0)\,[R_1 + R_2] + \frac{\omega R_1^2 C_1 I}{\omega^2 R_1^2 C_1^2 + 1}$$

$$\boxed{k = \frac{\omega R_1^2 C_1 I}{\omega^2 R_1^2 C_1^2 + 1}}$$

so,

$$\boxed{v(t) = \left[\frac{\omega R_1^2 C_1 I}{\omega^2 R_1^2 C_1^2 + 1}\right] e^{-\frac{t}{R_1 C_1}} + \left(\frac{I R_1}{\omega^2 R_1^2 C_1^2 + 1}\right)\sin(\omega t) - \left(\frac{\omega R_1^2 C_1 I}{\omega^2 R_1^2 C_1^2 + 1}\right)\cos(\omega t)}$$

Ans 2.



→ let $i_2(t)$ pass through R_1

KVL:

$$\boxed{R_2\, i(t) + L\frac{di(t)}{dt} = v(t)} \quad \text{—(1)}$$

KCL:

$$\boxed{i(t) + C\frac{dv(t)}{dt} = i_2(t)} \quad \text{—(2)}$$

KVL on left loop:

$$V_s = R_1 i_2(t) + v(t)$$

$$\boxed{V_s = R_1\left[i(t) + C\frac{dv(t)}{dt}\right] + v(t)} \quad \text{—(3)}$$

substitute (1) in (3)

$$V_s = R_1 i(t) + R_1 C\left[R_2\frac{di(t)}{dt} + L\frac{d^2 i(t)}{dt^2}\right] + R_2 i(t) + L\frac{di(t)}{dt}$$

$$V_s = R_1 CL\frac{d^2 i(t)}{dt^2} + (R_1R_2C + L)\frac{di(t)}{dt} + (R_1+R_2)\, i(t)$$

$$\boxed{\frac{d^2 i(t)}{dt^2} + \left(\frac{R_2}{L} + \frac{1}{R_1C}\right)\frac{di(t)}{dt} + \left(\frac{R_1+R_2}{R_1CL}\right) i(t) = \frac{V_s}{R_1CL}}$$

(a) any value for which:

$$\boxed{\left(\frac{R_2}{L} + \frac{1}{R_1C}\right)^2 < 4\left(\frac{R_1+R_2}{R_1CL}\right)}$$

(b) any value for which:

$$\boxed{\left(\frac{R_2}{L} + \frac{1}{R_1C}\right)^2 > 4\left(\frac{R_1+R_2}{R_1CL}\right)}$$

simplifies to

$$\boxed{L^2 > (\text{or} <)\ R_1^2R_2^2C^2(4L^2-1) + R_1R_2CL(4R_1^2CL-2)}$$

(—)

Ans 3:

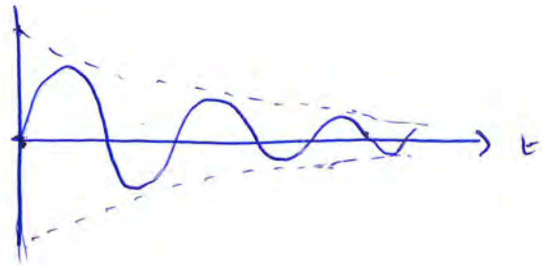
(a) $\boxed{s^2 + 4s + 10 = 0}$

(b) $b^2 - 4ac = 16 - 40$

$= -24$

< 0

Since $b^2 - 4ac < 0 \rightarrow$ $\boxed{\text{under-damped}}$ or $\boxed{\text{roots are complex conjugates}}$



(c) $s^2 + 4s + 10 = 0$

$s = \dfrac{-4 \pm \sqrt{-24}}{2}$

$\boxed{s = -2 \pm 2.45j}$

$\boxed{i(t) = e^{-2t}\left[k_1 \sin(2.45t) + k_2 \cos(2.45t)\right]}$

$i(0) = \boxed{k_2 = 1}$

$\dfrac{di(t)}{dt} = \left[-2k_1 \sin(2.45t) - 2k_2 \cos(2.45t) + 2.45k_1 \cos(2.45t) - 2.45k_2 \sin(2.45t)\right] e^{-2t}$

$\dfrac{di(0)}{dt} = -2k_2 + 2.45k_1 = 2$ $\quad \therefore k_2 = 1$

$2.45 k_1 = 4$

$\boxed{k_1 = 1.63}$

$$\boxed{i(t) = e^{-2t}\left[1.63 \sin(2.45t) + \cos(2.45t)\right]}$$

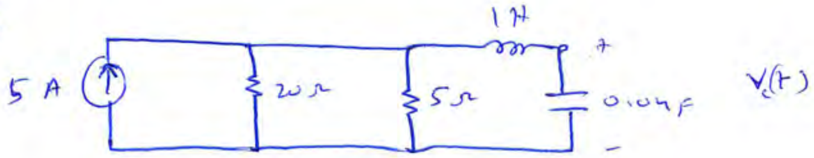
$\leftrightarrow$

Ans 4. $t<0$, capacitor oc $\boxed{i(0^-) = 0}$

$$v(0^-) = \frac{5}{30} \times 100$$

$$\boxed{v(0^-) = \frac{50}{3}}$$

for $t>0$



$$20 /\!/ 5 = \boxed{4\,\Omega}$$

RLC series circuit

KVL:

$$4\,i(t) + L\frac{di(t)}{dt} + \frac{1}{c}\int i(t)\,dt = 0$$

take derivative:

$$\boxed{\frac{d^2 i(t)}{dt} + 4\frac{di(t)}{dt} + 25\,i(t) = 0}$$

$$s^2 + 4s + 25 = 0$$

$$\boxed{s = -2 \pm \sqrt{21}\,j} \rightarrow \text{under-damped}$$

$$\boxed{i(t) = e^{-2t}\left[k_1 \sin(\sqrt{21}\,t) + k_2 \cos(\sqrt{21}\,t)\right]}$$

$$i(0) = \boxed{k_2 = 0}$$

$$v(t) = \frac{1}{c}\int i(t)\,dt \rightarrow \text{do cyclic integration}$$

$$v(t) = \underbrace{20}_{\substack{\text{at } t=\infty \\ v(t)=20}} + \frac{1}{c}\int i(t)\,dt$$

$v(t)=20$ → so this term caters for it.

$$\boxed{v(t) = 20 + e^{-2t}\left[(\sqrt{21}\,k_2 - 2k_1)\sin(\sqrt{21}\,t) + (-2k_2 - \sqrt{21}\,k_1)\cos(\sqrt{21}\,t)\right]}$$

$$v(0^-) = 20 - 2k_2 - \sqrt{21}\,k_1 = \frac{50}{3} \quad \therefore k_2 = 0 \quad \boxed{k_1 = 0.727}$$

$$\boxed{i(t) = 0.727\,e^{-2t}\sin(\sqrt{21}\,t)}$$

Q:5

(a) $2\frac{d^3i}{dt^3} + 9\frac{d^2i}{dt^2} + 13\frac{di}{dt} + 6i = 0$

Substituting $\frac{di}{dt} = s$.

$\Rightarrow \quad 2s^3 + 9s^2 + 13s + 6 = 0$

By either solving through long division / synthetic division / calculator.

$s = -2, -1, -3/2$

$\therefore \boxed{i(t) = k_1 e^{-2t} + k_2 e^{-t} + k_3 e^{-\frac{3}{2}t}}$

Now solving for k_1, k_2 and k_3.

At $t = 0^+$:

$i(0^+) = k_1 e^{-2(0^+)} + k_2 e^{-(0^+)} + k_3 (e)^{-\frac{3}{2}(0^+)}$

$\boxed{0 = k_1 + k_2 + k_3} \quad - (1)$

$\frac{di(t)}{dt} = -2k_1 e^{-2t} - k_2 e^{-t} - \frac{3}{2}k_3 e^{-\frac{3t}{2}}$

At $t = 0^+$:

$\frac{di(0^+)}{dt} = -k_1 (e)^{-(0^+)} - k_2 (e)^{-(0^+)} - 6k_3 (e)^{-6(0^+)}$

$\boxed{1 = -2k_1 - k_2 - \frac{3}{2}k_3} \quad - (2)$

$$\frac{d^2 i(t)}{dt} = 4k_1 e^{-t} + k_2 e^{-t} + \frac{9}{4} k_3 e^{-6t}$$

At $t = 0^+$:

$$\frac{d^2 i(0^+)}{dt} = 4k_1 e^{-(0^+)} + k_2 (e)^{-(0^+)} + \frac{9}{4} k_3 e^{-6(0^+)}$$

$$\boxed{-1 = 4k_1 + k_2 + \frac{9}{4} k_3} \text{ --- } (3)$$

Solving 1 , 2 and 3 simultaneously.

$k_1 = 3$

$k_2 = 5$

$k_3 = -8$

$$\Rightarrow \boxed{i(t) = 3e^{-2t} + 5e^{-t} - 8e^{-3/2 t}}$$

(b) $\frac{d^2 i}{dt^2} + 3\frac{di}{dt} + 2i = 10\sin 10t$ $\quad - (1)$

Non homogenous differential equation:

$\Rightarrow \quad s^2 + 3s + 2 = 0$

$s = -1, -2$

$i_c(t) = k_1 e^{-t} + k_2 e^{-2t}$

$i_p(t) = A \sin 10t + B \cos 10t$

Substituting i_p in (1):

$-100A \sin 10t - 100B \cos 10t + 30A \cos 10t - 30B \sin 10t$

$+ 2A \sin 10t + 2B \cos 10t = 10 \sin 10t$

Equating Coefficients:

for sin:

$-100A - 30B + 2A = 10$

$\boxed{-98A - 30B = 10}$ $\quad - (x)$

for cos:

$-100B + 30A + 2B = 0$

$\boxed{-98B + 30A = 0}$ $\quad - (y)$

Solving for (x) and (y):

$A = -0.094 \quad , B = 0.306A$

$\Rightarrow \; i_p(t) = -0.094 \sin 10t + (-0.029) \cos 10t$

Solving for k_1 and k_2:

At $t = 0^+$:

$$i(0^+) = k_1 e^{0^+} + k_2 e^{0^+} - 0.0094 \sin(0^+) - 0.029 \cos(0^+)$$

$$\boxed{1.029 = k_1 + k_2} -$$

$$\frac{di(0^+)}{dt} = -k_1 e^{0^+} - 2k_2 e^{0^+} - 0.094(10)\cos(0^+) + 0.029(10)\sin 10(0^+)$$

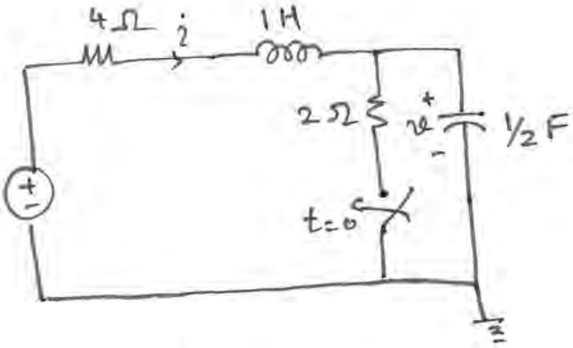
$$\boxed{k_1 + k_2 = 0.06}$$

Solving for k_1 and k_2 further:

$k_1 = 1.998$; $k_2 = -0.969$.

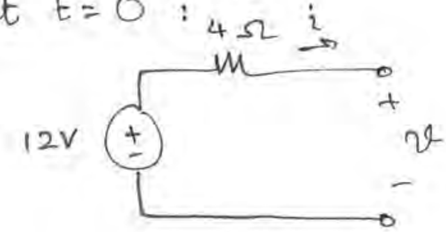
$$\Rightarrow i(t) = 1.998 e^{-t} - 0.969 e^{-2t} - 0.094 \sin 10t - 0.029 \cos 10t$$

Q:6

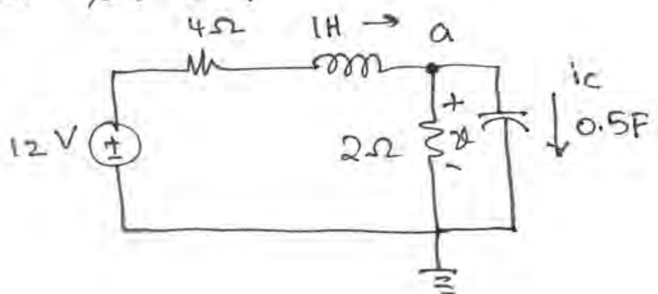


$$v_c(0^-) = 12V \quad ; \quad i(0^-) = 0\ A$$

at $t=0^-$:



at $t = 0^+$:



$\therefore v(0^+) = v(0^-) = 12V$

$i(0^+) = i(0^-) = 0\ A$

$i_c = \dfrac{C\,dv_c}{dt}$; Nodal at a for $t = 0^+$:

$$i(0^+) = i_c(0^+) + \frac{v_c(0^+)}{2}$$

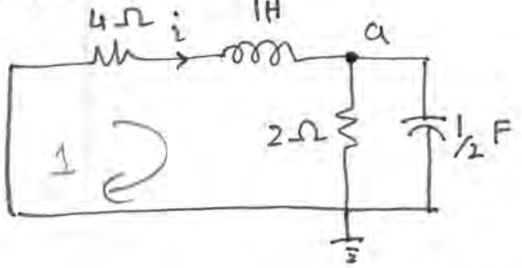
$$0 = i_c(0^+) + \frac{12}{2} \Rightarrow i_c(0^+) = -6A.$$

$$\therefore \frac{dv(0^+)}{dt} = \frac{-6}{0.5} = -12\ V/s$$

At $t=\infty$, capacitor will be open circuit and inductor short circuit.

$$i(\infty) = \frac{12}{4+2} = 2A \Rightarrow v(\infty) = 2i(\infty) = 4V$$

For $t > 0$:



$$i(t) = \frac{v(t)}{2} + \frac{1}{2}\frac{dv}{dt} \quad -②$$

Applying KVL to the left mesh (loop 1) results in

$$4i + 1\frac{di}{dt} + V = 0 \quad -①$$

Substituting ② in ① :

$$2v + 2\frac{dv}{dt} + \frac{1}{2}\frac{dv}{dt} + \frac{1}{2}\frac{d^2v}{dt^2} + v = 0$$

or

$$\frac{d^2v}{dt^2} + 5\frac{dv}{dt} + 6v = 0$$

Substituting $\frac{dv}{dt} = s$

$$\Rightarrow \quad s^2 + 5s + 6s = 0$$

$$s = -2, -3$$

$$\Rightarrow \quad v(t) = K_1 e^{-2t} + K_2 e^{-3t}$$

At steady state : $v(\infty) = 4$.

$$\Rightarrow \quad v(t) = v(\infty) + v(t) = 4 + K_1 e^{-2t} + K_2 e^{-3t}$$

at $t = 0^+$: $\cancel{v(0^+)}$ $12 = 4 + K_1 + K_2 \Rightarrow K_1 + K_2 = 8 \quad -(x)$

$$\frac{dv}{dt} = -2K_1 e^{-2t} - 3K_2 e^{-3t}$$

at $t = 0^+$:

$$-12 = -2k_1 - 3k_2$$

$$\Rightarrow 2k_1 + 3k_2 = 12 \quad - (Y)$$

Solving (X) & (Y) simultaneously yields:

$$K_1 = 12 \quad ; \quad k_2 = -4.$$

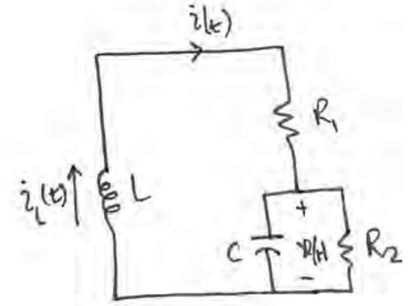
$$\Rightarrow v(t) = 4 + 12e^{-2t} - 4e^{-3t}$$

Now solving for i for $t > 0$:

$$i = \frac{v}{2} + \frac{1}{2}\frac{dv}{dt} = 2 + 6e^{-2t} - 2e^{-3t} - 12e^{-2t} + 6e^{-3t}$$

$$= 2 - 6e^{-2t} + 4e^{-3t} \quad A$$

Q:7



$R_1 = 10\Omega$; $R_2 = 8\Omega$; $C = \frac{1}{8}F$;

$L = 2H$, $v_c(0) = 1V$; $i_L(0^+) = 0.5A$

(a) $L\frac{di}{dt} + R_1 i(t) + v(t) = 0$

(b) $C\frac{dv}{dt} + \frac{v(t)}{R_2} = i(t)$

Substituting b into a :

$$\frac{d^2v}{dt^2} + \left[\frac{1}{R_2 C} + \frac{R_1}{L}\right]\frac{dv}{dt} + \frac{R_1 + R_2}{R_2 L C} v = 0$$

Substituting the given values , yields :

$$\frac{d^2v}{dt^2} + 6\frac{dv}{dt} + 9v = 0$$

Substituting $\frac{dv}{dt} = s$

$\Rightarrow$ $s^2 + 6s + 9 = 0$

$s = -3, -3$.

This becomes the critically damped case , hence :

$$\boxed{v(t) = k_1 e^{-3t} + k_2 t e^{-3t}} \quad -①$$

At $t=0$:

$$\boxed{v(0) = v_c(0) = 1 = k_1}$$

Differenting ① :

$$\frac{dv}{dt} = -3k_1 e^{-3t} + k_2 e^{-3t} - 3k_2 t e^{-3t} \quad -②$$

$$i_c(t) = C\frac{dv_c}{dt}$$

$$\Rightarrow \frac{dv_c}{dt} = \frac{i(t)}{C} - \frac{v(t)}{R_2C} \quad -(3)$$

Putting (2) = (3):

At $t=0$:

$$\boxed{3 = -3k_1 + k_2}$$

Substituting $k_1 = 1$, yields:

$$\boxed{k_2 = 6}$$

$$\Rightarrow \boxed{v(t) = e^{-3t} + 6te^{-3t}} \quad V$$

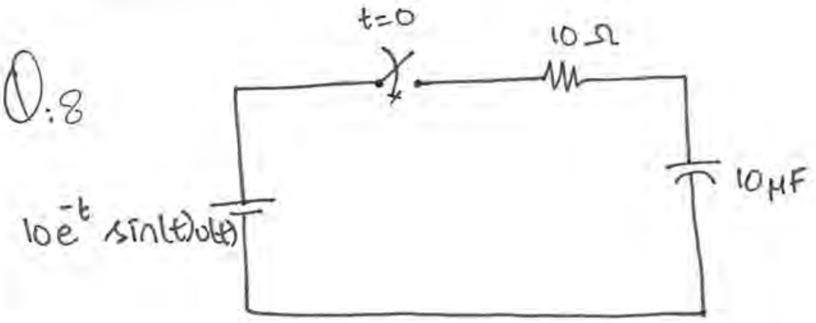
Now we'll put $v(t)$ into $i(t)$ (b):

$$i(t) = C\frac{dv_c(t)}{dt} + \frac{v(t)}{R_2}$$

$$i(t) = \frac{1}{8}\left[-3e^{-3t} + 6e^{-3t} - 18te^{-3t}\right] + \frac{1}{8}\left[e^{-3t} + 6te^{-3t}\right]$$

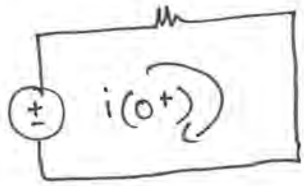
$$\Rightarrow \boxed{i(t) = \tfrac{1}{2}e^{-3t} - \tfrac{3}{2}te^{-3t}} \quad A$$

Q:8



$v_c(0^-) = v_c(0^+) = 0V$

$i(0^+) = \dfrac{10e^{-(0^+)}\sin(0^+)}{10}$



$\boxed{i(0^+) = 0\ A}$

For $t \geq 0$;

Applying loop analysis:

$$10i(t) + \frac{1}{C}\int i(t) = 10e^{-t}\sin(t)u(t)$$

$$i(t) + 10000\int i(t) = e^{-t}\sin(t)$$

Differentenliating w.r.t ~~ts~~ t :

$$\frac{di(t)}{dt} + 10000i = e^{-t}\cos(t) - e^{-t}\sin(t) \quad \text{—①}$$

This is a non homogenous equation, hence:

~~Sace~~

$$si + 10^4 i = 0$$

$$\Rightarrow s = -10^4$$

$$i_c = ke^{-10,000t}$$

The particular solution will be :

$$i_p(t) = [A\cos t + B\sin t]e^{-t}$$

Putting the particular integral into the equation (1):

$$-Ae^{-t}\sin t - Ae^{-t}\cos t + Be^{-t}\cos t - Be^{-t}\sin t$$
$$+ 10{,}000e^{-t}A\cos t + 10{,}000\,Be^{-t}\sin t = e^{-t}\cos t - e^{-t}\sin t$$

Equating coefficients of $e^{-t}\sin t$:

$$-A - B + 10{,}000B = -1$$

$$\boxed{-A + 9999B = -1} \quad - (a)$$

Equating coefficients of $e^{-t}\cos t$:

$$-A + B + 10{,}000A = 1$$

$$\boxed{B + 9999A = 1} \quad - (b)$$

Solving (a) and (b) simultaneously ; yields :

$$A = 10^{-4}, \quad B = -10^{-4}$$

$$\Rightarrow \quad i(t) = i_c(t) + i_p(t)$$

$$= K e^{-10{,}000t} + 10^{-4}e^{-t}\cos t - 10^{-4}e^{-t}\sin t$$

Now solving for k:

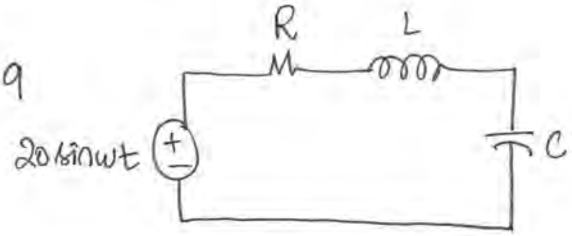
At $t = 0^+$:

$$i(0^+) = k e^{-10,000(0^+)} + \left[10^{-4}\cos(0^+) - 10^{-4}\sin(0^+)\right] e^{-(0^+)}$$

$$\boxed{k = -10^{-4}}$$

$$\Rightarrow \boxed{i(t) = -10^{-4} e^{-10,000t} + 10^{-4} e^{-t}\cos t - 10^{-4} e^{-t}\sin wt}$$

Q:9



$R = 2\,\Omega$

$L = 1 \times 10^{-3}\,H$

$C = 0.4 \times 10^{-6}\,F$

(a) The resonant frequency:

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1\times10^{-3} \times 0.4\times10^{-6}}} = 50\times10^{3}\ rad/s$$

(b) The bandwidth will be:

$$B = R/L = \frac{2}{1\times10^{-3}} = 2\times10^{3}\ rad/s$$

(c) The quality factor will be:

$$Q = \frac{\omega_0}{B} = \frac{50\times10^{3}}{2\times10^{3}} = 25.$$