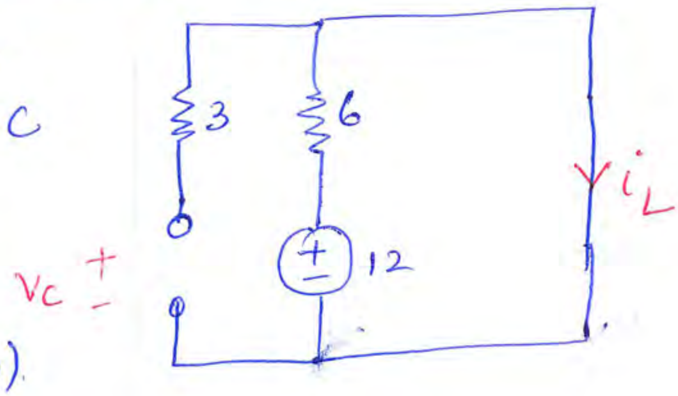
Second-Order Circuits

SOLUTIONS

Problem 01

* First analyze circuit at $t=0^-$

- Inductor ; SC
- Capacitor ; OC



$v_C(0^-) = 0V = V_C(0)$

$i_L(0^-) = \frac{12}{6} = 2A = i_L(0)$

$i(0^-) = 0A$

$\Rightarrow \boxed{i(0) = 2A}$ $\quad i(0) = i_L = 2A$

Circuit is simple RLC series at $t=0$.

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = 0 \quad \text{— (01)}$$

$V_C(0)=0$

At $t=0$

$$L\frac{di}{dt}(0) + Ri(0) + 0 = 0$$

$$\Rightarrow \frac{di}{dt}(0) = \frac{-(2)(3)}{2} = \boxed{-3\ A/sec}$$

* Solve for $i(t)$

Take derivative of (01)

$$\Rightarrow L\frac{d^2 i(t)}{dt^2} + R\frac{di}{dt} + \frac{i(t)}{C} = 0$$

$$\Rightarrow \frac{d^2 i}{dt^2} + \frac{3}{2}\frac{di}{dt} + \frac{i(t)}{2} = 0$$

Characteristic Equation :-

$$s^2 + \frac{3}{2}s + \frac{1}{2} = 0 \Rightarrow s = -1, -\frac{1}{2}$$

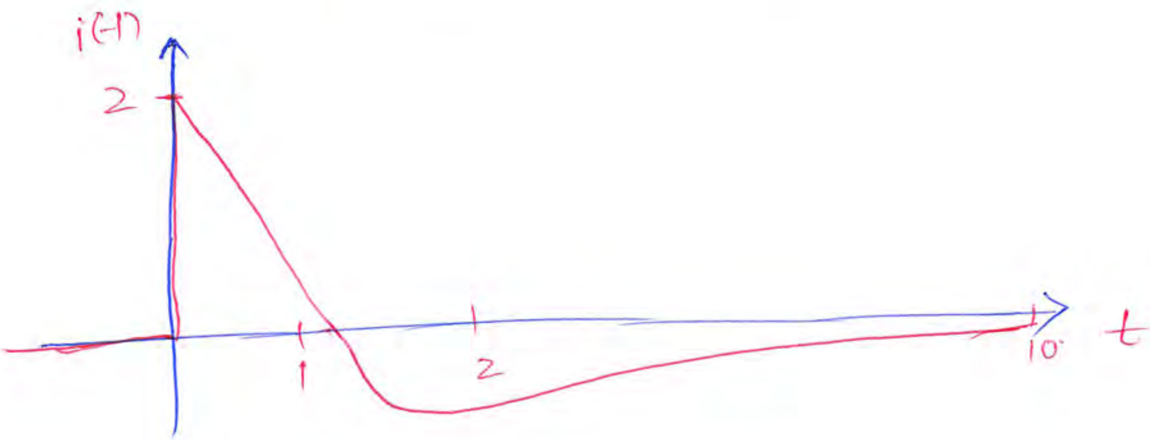
$$\Rightarrow i(t) = K_1 e^{-t} + K_2 e^{-t/2}$$

$$i(0) = 2 = K_1 + K_2$$

$$\frac{di}{dt}(0) = -3 = -K_1 - \frac{K_2}{2}$$

$$\Rightarrow \boxed{K_2 = -2} \Rightarrow \boxed{K_1 = 4}$$

$$i(t) = \begin{cases} 0 \text{ A} & t < 0 \\ 2 \text{ A} & t = 0 \\ 4e^{-t} - 2e^{-t/2} \text{ A} & t \geq 0 \end{cases}$$



Problem 02

* Analyze circuit at $t = 0^-$

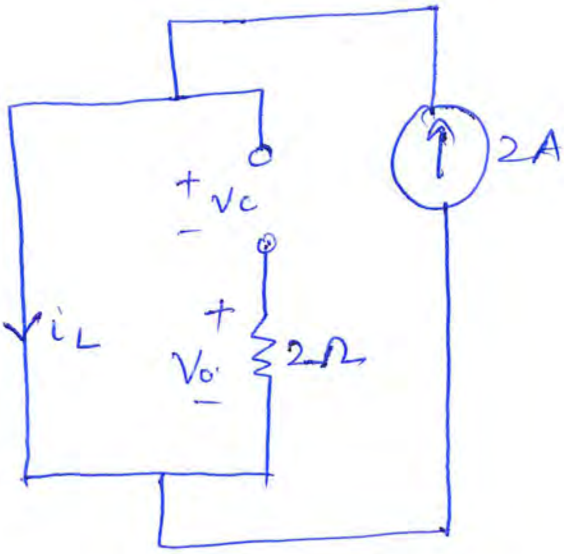
Inductor : SC

Capacitor : OC

$v_c(0^-) = 0 = v_c(0^+)$

$i_L(0^-) = 2 = i_L(0^+)$

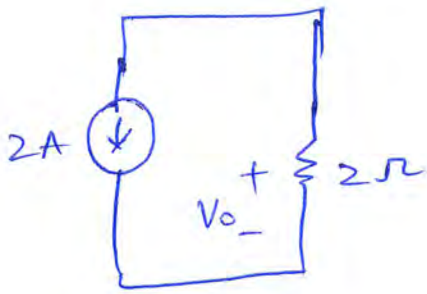
$v_o(0) = (-2)(2) = \boxed{-4V}$



⇒ We have series RLC at $t = 0$;

$$\Rightarrow L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = 0$$

$t = 0^+$



$$\Rightarrow \frac{di}{dt}(0^+) = -\frac{R}{L}\,i(0^+)$$
$$= -(4)(2) = -8A/sec$$

* Here $i(t)$ is taken as the current in the counter-clockwise direction, that is, $i = i_L$

$\because V_o = -iR$

$$\frac{dv_o}{dt}(0^+) = -\frac{di}{dt}(0^+)\,(R) = \boxed{+16V/sec}$$

* Solve for $i(t)$ or $\boxed{v_o(t) = -i(t)R}$

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i(t) = 0$$

$$\Rightarrow \quad -\frac{1}{R}\frac{d^2 v_o(t)}{dt^2} - \frac{1}{L}\frac{dv_o}{dt} + \frac{1}{RLC} v_o(t) = 0.$$

$$\Rightarrow \quad \frac{d^2 v_o(t)}{dt^2} + \frac{R}{L}\frac{dv_o}{dt} + \frac{1}{LC} v_o(t) = 0.$$

$$\Rightarrow \quad \frac{d^2 v_o}{dt^2} + 4\frac{dv_o}{dt} + 3 v_o(t) = 0.$$

$$\Rightarrow \quad s^2 + 4s + 3 = 0 \Rightarrow s_1 = -1,\ s_2 = -3$$

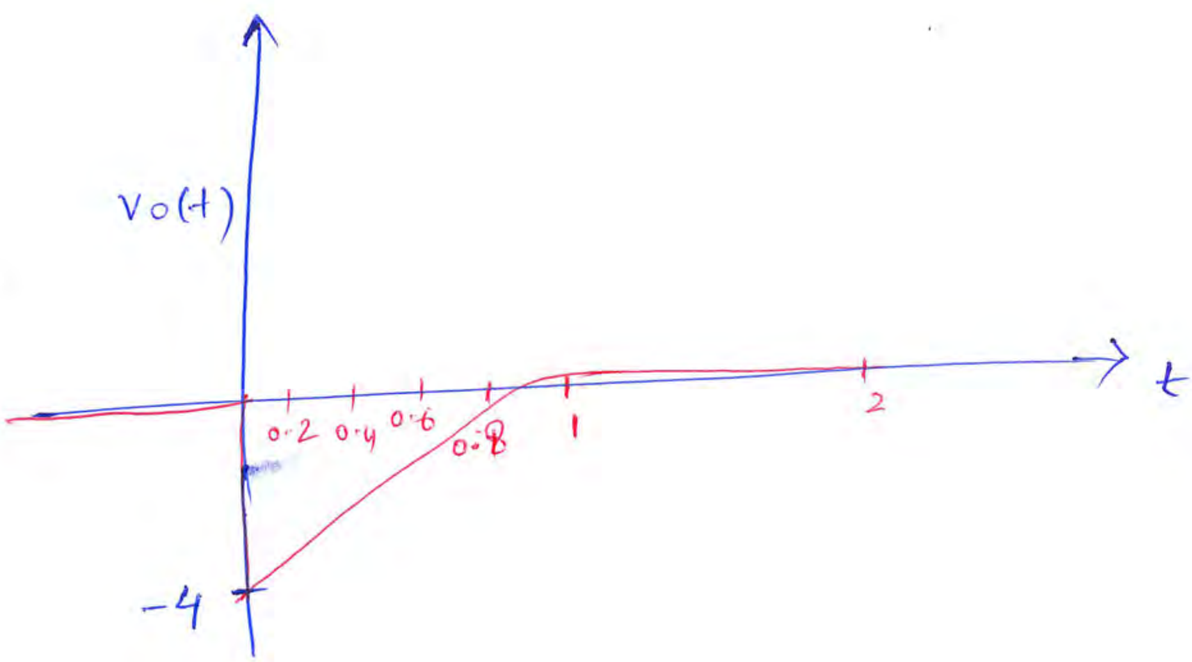
$$\Rightarrow \quad v_o(t) = K_1 e^{-t} + K_2 e^{-3t}$$

$$v_o(0) = -4 = K_1 + K_2$$

$$\frac{dv_o(0)}{dt} = 16 = -K_1 - 3K_2$$

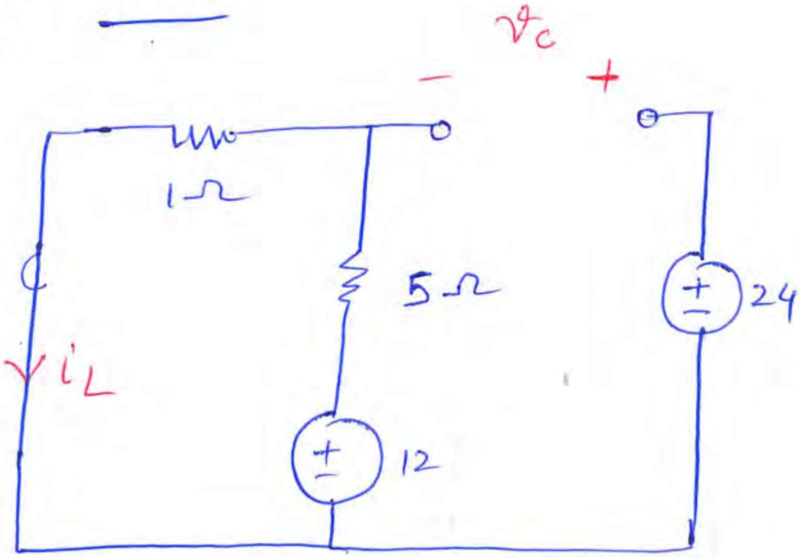
$$\Rightarrow \boxed{K_2 = -6},\ \boxed{K_1 = 2}$$

$$\Rightarrow \boxed{v_o(t) = 2e^{-t} - 6e^{-3t}\ V}$$



Problem 03

* Analyse Circuit at $\underline{t=0^-}$

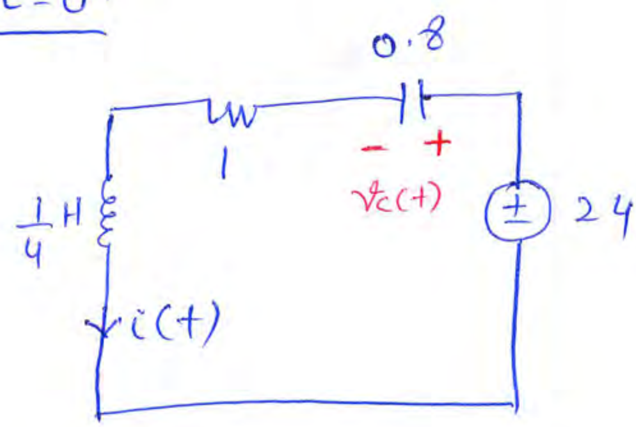


$i_L(0^-) = \boxed{2A}$

$v_c(0^-) = 24 - 2$
$= \boxed{22V}$
$= v_c(0^+)$

At $t=0^+$

$\underline{t=0^+}$



current through capacitor is i_L

$$\Rightarrow \quad i_L(0^+) = C\frac{dV_c}{dt}(0^+)$$

$$\Rightarrow \quad \frac{dV_c}{dt}(0^+) = \frac{i_L(0^+)}{C} = 2.5\ V/sec.$$

Solve for $i(t)$

$$\frac{1}{4}\frac{di}{dt} + i + \frac{1}{0.8}\int i = 24$$

$$\Rightarrow \quad \frac{d^2 i}{dt^2} + 4\frac{di}{dt} + 5i = 0$$

$$s^2 + 4s + 5 = 0 \Rightarrow s_1, s_2 = -2 \pm 1i$$

$$i(t) = e^{-2t}\left(K_1 \cos t + K_2 \sin t \right).$$

$$i(0) = \boxed{K_1 = 2}.$$

To find $\frac{di}{dt}(0^+)$.

$$\frac{1}{4}\frac{di}{dt}(0^+) + i(0^+) + \cancel{\frac{1}{0.1}}\, v_c(0^+) = 24$$

$$\frac{di}{dt}(0^+) = 4(24 - 24) = \boxed{0}$$

Use to find K_2

$$\Rightarrow \frac{di}{dt}(0) = e^{-2t}\left(-K_1 \sin t + K_2 \cos t\right) - 2e^{-2t}\left(K_1 \cos t + K_2 \sin t\right)\Big|_{t=0} = 0.$$

$$= K_2 - 2K_1 = 0 \Rightarrow \boxed{K_2 = 4}$$

$$\boxed{i(t) = e^{-2t}\left(2\cos t + 4 \sin t\right)}.$$

Now

$$v_c(t) = 24 - i(t) - \frac{1}{4}\frac{di}{dt}$$

$$= 24 - e^{-2t}\left(2\cos t + 1.5 \sin t\right)$$

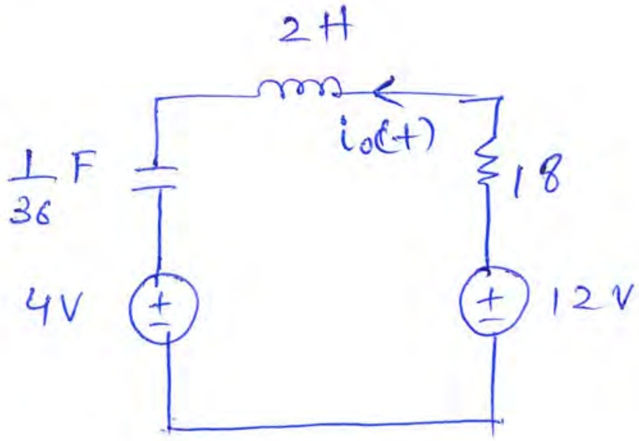
Problem 04

$t = 0^-$

$$i_L(0^-) = i_o(0^-) = \frac{12}{24} = 0.5A$$

$$V_c(0^-) = V_c(0^+) = 0V$$

$t = 0^+$



$$i_o(0^+) = 0.5A$$

$$18 i_o + L\frac{di_o}{dt} + \underbrace{36\int i_o dt}_{V_c(0^+)=0} = 8$$

$$\Rightarrow \frac{di_o}{dt}(0^+) = \frac{(8-9)}{2} = -0.5 A/sec.$$

Solve for i_o :

$$2\frac{d^2 i_o}{dt^2} + 18\frac{di_o}{dt} + 36 i_o = 0$$

$$\Rightarrow s^2 + 9s + 18 = 0$$

$$s_1 = -6, \quad s_2 = -3$$

$$\Rightarrow i_o(t) = K_1 e^{-3t} + K_2 e^{-6t}$$

$$K_1 + K_2 = 0.5$$

$$-3K_1 - 6K_2 = -0.5$$

$$\Rightarrow \boxed{K_1 = 5/6 \quad K_2 = -\frac{1}{3}}$$

Problem 05

$\boxed{6-25}$

* We first analyze circuit at $t = 0^-$
* Find $i(t)$ at $t = 0^-$
* Circuit is in steady state ; $\boxed{i(t) = i_p(t)}$

Equation for $i(t)$: $\boxed{L\frac{di}{dt} + Ri + \underbrace{\frac{1}{C}\int i}_{v_C} = 100\sin 377t}$ — (01)

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = (100)(377)\cos 377t$$

$$\Rightarrow \frac{d^2 i}{dt^2} + 10^3\frac{di}{dt} + (2\times10^6)i = (37700)\cos 377t \quad \text{— (02)}$$

$$i_p(t) = A\cos 377t + B\sin 377t$$

$\Rightarrow$ $\boxed{A = 0.0195}$ $\boxed{B = 0.0040}$ After substituting $i_p(t)$ in (02)

$$i(t) = (0.0195\cos 377t + 0.0040\sin 377t)\text{ Amps } t < 0$$

* Inductor ensures $i(0^+) = i(0^-) = \underline{0.0195}$ Amperes
* $v_C(0^-) = -R\,i(0) - L\left.\frac{di}{dt}\right|_{t=0^-}$

$$= -10^3(0.0195) - (1)(377)(0.0040)$$

$$= -21.0080 \text{ Volts}$$

Now analyze at $t = 0^+$

$$\boxed{\frac{di}{dt} + \underbrace{\frac{1}{LC}\int i}_{V_c} = 100\sin 377t}$$ 03

$$\frac{d^2 i}{dt^2} + \frac{i}{LC} = (100)(377)\cos 377t$$

$$\frac{d^2 i}{dt^2} + (2\times 10^6) = (37700)\cos 377t$$

$$i(t) = i_c(t) + i_p(t)$$

$$i_p(t) = A\cos 377t + B\sin 377t$$

$$\boxed{A = 0.0203, \quad B = 0}$$

$i_c(t) = ?$

$$s^2 + 2\times 10^6 s = 0 \Rightarrow s_1, s_2 = j1414.2, -j1414.2$$

$$\Rightarrow i_c(t) = K_1 \cos 1414.2t + K_2 \sin 1414.2t$$

Find K_1 and K_2?

$$i(0) = 0.0195 = K_1 + 0.0203 \Rightarrow \boxed{K_1 = -8\times 10^{-4}}$$

* To find K_2; we first find $\frac{di}{dt}(0^+)$ using 03

$$\frac{di}{dt}(0^+) = -V_c = 21.0080 \text{ A/sec}$$

$$377(B) + (1414.2)(K_2) = 21.0080 \Rightarrow \boxed{K_2 = 149\times 10^{-4}}$$

$$\Rightarrow i(t) = 0.0203\cos 377t + 10^{-4}\left[-8\cos 1414.2t + 149\sin 1414.2t\right] \text{A}$$