Department of Electrical Engineering
School of Science and Engineering

EE240 Circuits I - Fall 2020

ASSIGNMENT 5 – SOLUTIONS

Q 1. Analyse the circuit at $t = 0^-$,

$$V_c(0^-) = 10V$$

Since capacitor does not allow instantaneous change in voltage,

$$V_c(0^+) = 10V$$

Form a differential equation,

$$6i(t) + \frac{1}{C}\int i(t)dt = 40e^{-t}u(t)$$

Taking its derivative,

$$\frac{di(t)}{dt} + \frac{1}{0.331 \cdot 6}i(t) = -\frac{40}{6}e^{-t}u(t)$$

The forms of the particular and the complementary solution will be:
Complementary solution $i_c(t)$ (form a characteristic equation):

$$s + 0.5 = 0$$
$$s = -0.5$$

Hence,

$$i_c(t) = K_1 e^{-0.5t}$$

Particular solution $i_p(t)$ (will have the form of the excitation):

$$i_p(t) = K_2 e^{-t}$$

Lets start by finding the particular solution, substitute $i_p(t)$ in the differential equation,

$$\frac{di_p(t)}{dt} + 0.5i_p(t) = \frac{40}{6}e^{-t}$$
$$-K_2e^{-t} + 0.5K_2e^{-t} = \frac{-40}{6}e^{-t}u(t)$$
$$-0.5K_2e^{-t} = \frac{-40}{6}e^{-t}u(t)$$
$$K_2 = 13.33$$

So,

$$i(t) = K_1e^{-0.5t} + 13.33e^{-t}$$

Using the initial equation we get that,

$$V_c(t) = 40e^{-t} - 6i(t)$$

Substitute the expression for i(t),

$$V_c(t) = 40e^{-t} - 6(K_1e^{-0.5t} + 13.33e^{-t})$$

Solve at $t = 0^+$,

$$K_1 = -8.33$$

Finally,

$$i(t) = -8.33e^{-0.5t} + 13.33e^{-t}$$

Q 2. (a) Nodal analysis,

$$C\frac{dV_c}{dt} + \frac{V_c}{R} + \frac{1}{L}\int V_c dt = 0$$

For $\frac{dV_c(0^+)}{dt}$ evaluate equation at $t = 0^+$,

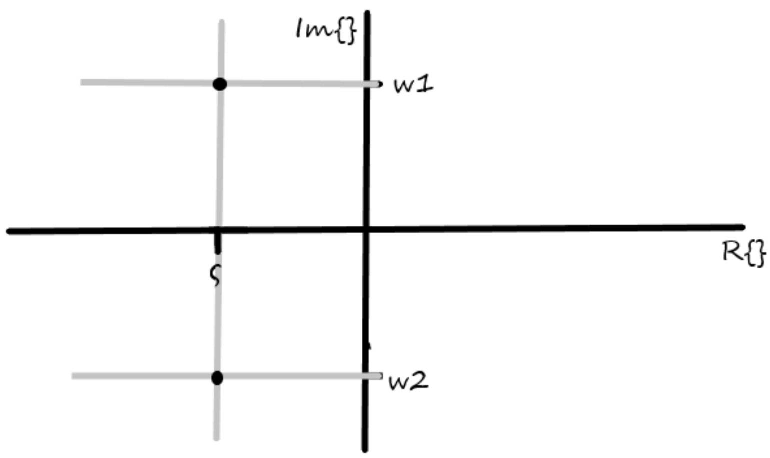
$$\frac{dV_c(0^+)}{dt} = -\frac{V_c}{RC} - \frac{I_L}{C}$$
$$= \frac{-12.4}{24 \cdot 0.5} - \frac{250m}{0.5} = -1.6Vs^{-1}$$

(b) Derivate the equation to get,

$$C\frac{d^2V_c}{dt^2} + \frac{1}{R}\frac{dV_c}{dt} + \frac{V_c}{L} = 0$$
$$\frac{d^2V_c}{dt^2} + \frac{1}{RC}\frac{dV_c}{dt} + \frac{V_c}{LC} = 0$$
$$\frac{d^2V_c}{dt^2} + \frac{1}{12}\frac{dV_c}{dt} + \frac{1}{5}V_c = 0$$

(c) Form a characteristic equation,

$$s^2 + \frac{1}{12}s + \frac{1}{5} = 0$$
$$s_1 = -0.04 + 0.44j$$
$$s_2 = -0.04 - 0.44j$$



(d)

(e)

$$V_c(t) = (K_1\cos(wt) + K_2\sin(\omega t))e^{\sigma t}$$

Where,

$$\sigma = -0.04$$
$$\omega = 0.44$$

(σ signifies the decay and ω is the frequency of oscillation)

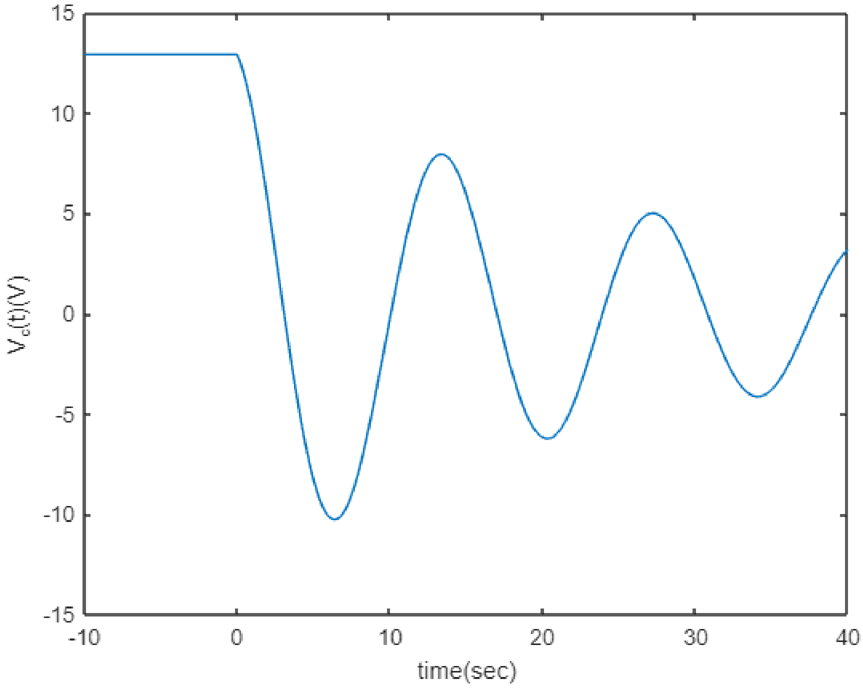
$$V_c(t) = e^{-0.04t}(K_1\cos(0.44t) + K_2\sin(0.44t))$$

$$\frac{dV_c(t)}{dt} = -0.04e^{-0.04t}(K_1\cos(0.44t) + K_2\sin(0.44t)) + e^{-0.04t}(-0.44K_1\sin(0.44t) + 0.44K_2\cos(0.44t))$$

Substitute $V_c(0^+)$ and $\frac{dV_c(0^+)}{dt}$ when evaluating these equations at $t = 0^+$,

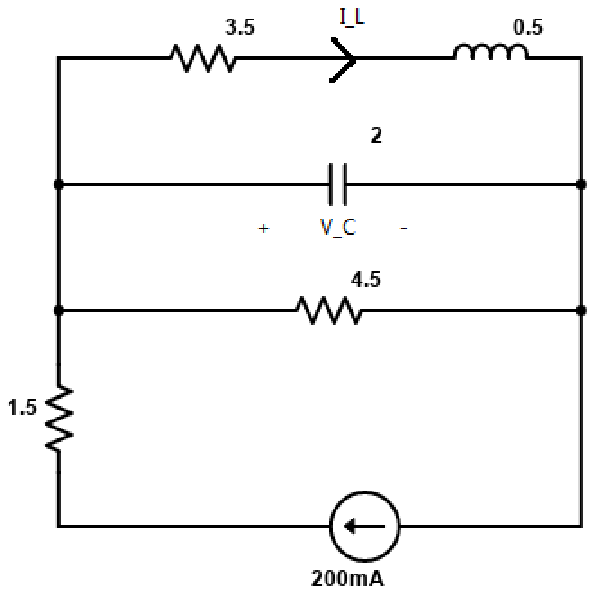
$$K_1 = 12.9$$
$$K_2 = -2.5$$

$$V_c(t) = \begin{cases} e^{-0.04t}(12.9\cos(0.44t) - 2.5\sin(0.44t))V & t \geq 0 \\ 12.9V & t < 0 \end{cases}$$



MATLAB code:
syms t
y = piecewise($t < 0$, 12.9, $t >= 0$, $exp(-0.04.*t).*(12.9.*cos(0.44.*t)) - 2.5.*sin(0.44.*t))$
fplot(y)
axis([-10 40 -15 +15])
xlabel('time(sec)')
ylabel($'V_c(t)(V)'$)

Q 3. (a) At $t = 0^-$,



For V_c, notice that the capacitor is in parallel with 3.5Ω and 4.5Ω resistors. The 1.5Ω resistor is redundant as it is in series with the current source.

$$V_c = IR$$

$$(200 \cdot 10^{-3}) \cdot \frac{3.4 \cdot 4.5}{8} = 0.39V$$

For I_L, use current divider or simply ohm's law,

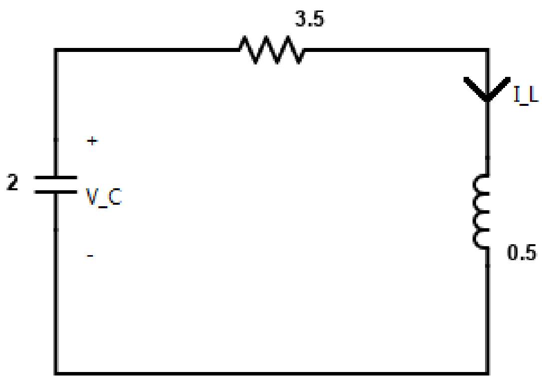
$$V_c = I_L R$$

$$I_L = \frac{V_c}{R} = \frac{0.39}{3.5} = 0.11A$$

So,

$$V_c(0^-) = 0.39V, \quad I_L(0^-) = 0.11A$$

(b) At $t = 0^+$,



Loop equation:

$$-\frac{1}{C}\int I_L dt + I_L R + L\frac{dI_L}{dt} = 0 \tag{1}$$

For $\frac{dI_L(0^+)}{dt}$,
Evaluate (1) at 0^+, (note that $V_c(0^-) = V_c(0^+)$ and $I_L(0^-) = I_L(0^+)$),

$$-0.39 + (0.11 \cdot 3.5) + 0.5\frac{dI_L(0^+)}{dt} = 0$$

$$\frac{dI_L(0^+)}{dt} = 0.01As^{-1}$$

(c) Take derivative of (1),

$$\frac{I_L}{C} + R\frac{dI_L}{dt} + L\frac{d^2I_L}{dt^2} = 0$$

$$\frac{d^2I_L}{dt} + 7\frac{dI_L}{dt} + I_L = 0 \tag{2}$$

$$s^2 + 7s + 1 = 0$$
$$s_1 = -0.15$$
$$s_2 = -6.85$$

Notice that the roots are real and distinct (Over-damped),

$$I_L(t) = K_1e^{-0.15t} + K_2e^{-6.85t}$$

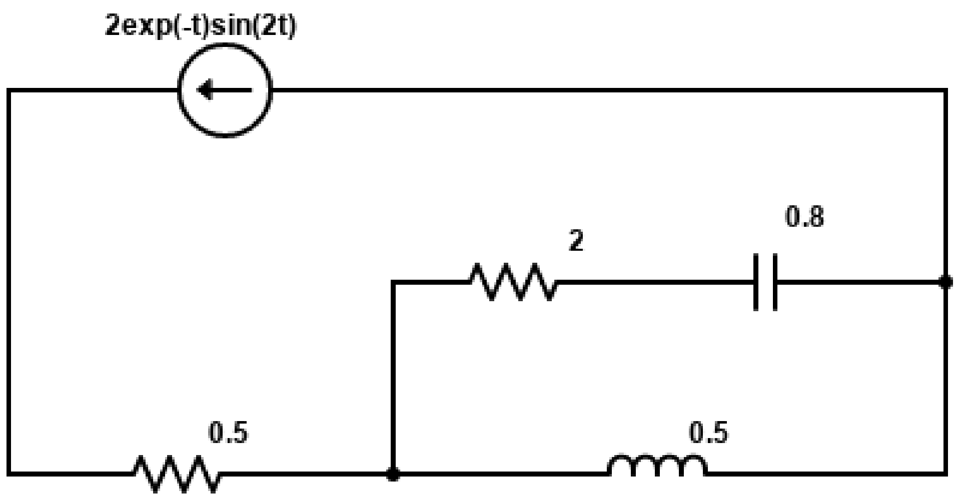
Use $I_L(0^+)$ and $\frac{dI_L(0^+)}{dt}$ to find K_1 and K_2,

$$K_1 = 0.11 \quad K_2 = -0.004$$

So

$$I_L(t) = \begin{cases} 0.11e^{-0.15t} - 0.004e^{-6.85t}A & 0 \leq t < 4 \\ 0.06A & t = 4 \end{cases}$$

(d) For $t > 4$



0.5Ω is redundant as it is in series to the current source.
Loop equation:

$$0.5\frac{dI_L}{dt^2} + 2(I_L - 2e^{-t}\sin(2t)) + \frac{1}{0.8}\int(I_L - 2e^{-t}\sin(2t)) = 0 \tag{3}$$

$$\frac{d^2I_L}{dt^2} + 4\frac{dI_L}{dt} + 2.5I_L = 3e^{-t}\sin(2t) - 16e^{-t}\cos(2t) \tag{4}$$

For $\frac{dI_L(4^+)}{dt}$, Evaluate (3) at t=4,

$$\frac{dI_L(4^+)}{dt} = 2(-2I_L(4) + 4e^{-4}sin(8) + 0)$$
$$= -0.22As^{-1}$$

(e) Refer to Equation (4)

(f)

$$y_p(t) = [A\cos(2t) + B\sin(2t)]e^{-t}$$
$$\frac{dy_p(t)}{dt} = e^{-t}[-2A\sin(2t) + 2B\cos(2t)] - e^{-t}[A\cos(2t) + B\sin(2t)]$$
$$\frac{d^2y_p(t)}{dt^2} = 4Ae^{-t}\sin(2t) - 3Ae^{-t}\cos(2t) - 4Be^{-t}\cos(2t) - 3Be^{-t}\sin(2t)$$

Substitute in (4),

$$\sin(2t)e^{-t}[4A - 3B - 8A - 4B + 2.5B] + \cos(2t)e^{-t}[-3A - 4B + 8B - 4A + 2.5A] = 3e^{-t}\sin(2t) - 16e^{-t}\cos(2t)$$

So,

$$A = 1.66$$
$$B = -2.14$$

$$I_p(t) = [1.66\cos(2t) - 2.14\sin(2t)]e^{-t}$$

(g)

$$s^2 + 4s + 2.5 = 0$$
$$s_1 = -0.78$$
$$s_2 = -3.22$$

Roots are real and distinct(over-damped).

$$I_c(t) = K_1e^{-0.78t} + K_2e^{-3.22t}$$

Combine the complementary and particular solution to find K_1 and K_2,

$$I_L(t) = I_c(t) + I_p(t)$$
$$I_L(t) = K_1e^{-0.78t} + K_2e^{-3.22t} + [1.66\cos(2t) - 2.14\sin(2t)] - e^{-t}[1.66\cos(2t) - 2.14\sin(2t)]$$

Use $I_L(4^+)$ and $\frac{dI_L(4+)}{dt}$ to find K_1 and K_2,

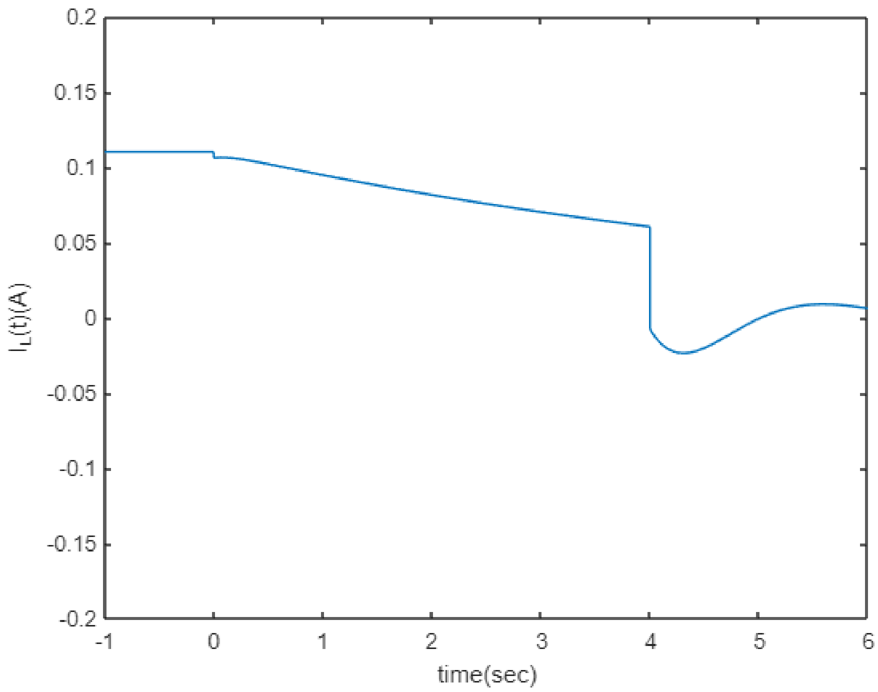
$$K_1 = -0.03$$
$$K_2 = 14455$$

$$I_c(t) = -0.03e^{-0.78t} + 14455e^{-3.22t}A$$

(h)

$$I_c(t) = \begin{cases} 0.11A & t < 0 \\ 0.11e^{-0.15t} - 0.004e^{-6.85t}A & 0 \leq t \leq 4 \\ -0.03e^{-0.78t} + 14455e^{-3.22t} + [1.66cos(2t) - 2.14sin(2t)]e^{-t}A & t > 4 \end{cases}$$

Plot:



Matlab code:

```
syms t
y = piecewise(t < 0, 0.11, t <= 4, 0.11.*exp(-0.15.*t) - 0.004.*exp(-6.85.*t), t > 4,
-0.03.*exp(-0.78.*t)+14455.*exp(-3.22.*t)+(1.66.*cos(2.*t)-2.14.*sin(2.*t)).*exp(-t))
fplot(y)
axis([-1 6 -0.2 +0.2])
xlabel('time(sec)')
ylabel('I_L(t)(A)')
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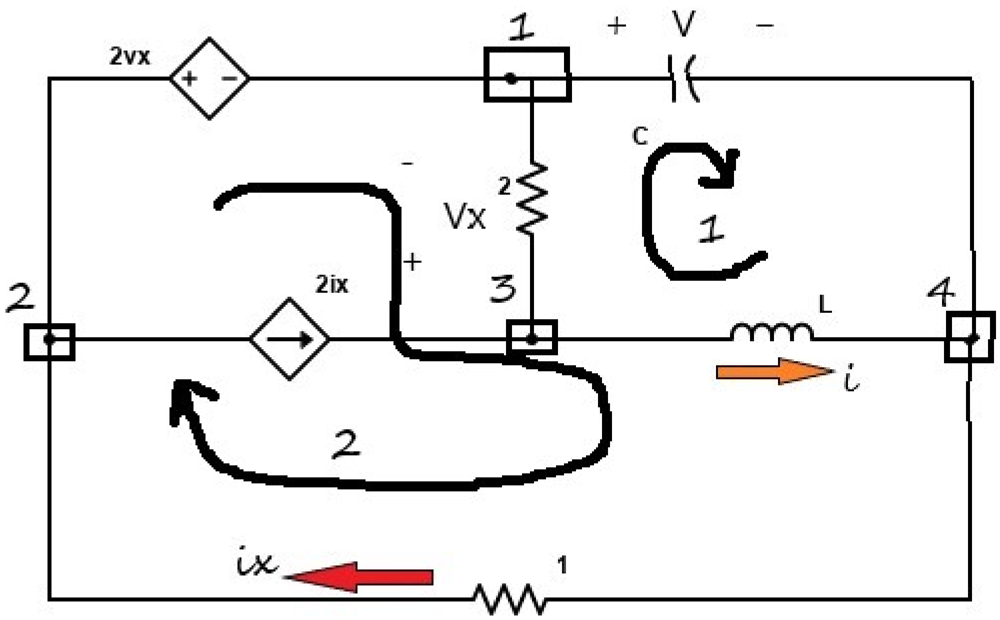
Q 4. Information given:

$$v(0) = 10V$$
$$i(0) = 0$$
$$L = 0.1H$$
$$C = 0.01F$$



Using KCL, write currents at node 2:

$$i_x + i_y = 2i_x$$
$$i_y = i_x$$

Write nodal equation at node 1:

$$i_c - \frac{v_x}{2} + i_y = 0$$
$$C\frac{dv}{dt} - \frac{v_x}{2} + i_x = 0$$
$$0.01\frac{dv}{dt} - \frac{v_x}{2} + i_x = 0 \rightarrow (1)$$

Write nodal equation at node 3:

$$i - 2i_x + \frac{v_x}{2} = 0$$
$$2i - 4i_x + v_x = 0$$
$$v_x = 4i_x - 2i \rightarrow (2)$$

Substitute (2) in (1),

$$0.01\frac{dv}{dt} + i - i_x = 0 \rightarrow (3)$$

Write KVL around loop 1:

$$v - v_L + v_x = 0$$
$$v - L\frac{di}{dt} + 4i_x - 2i = 0$$
$$\frac{di}{dt} - 10v - 40i_x + 20i = 0 \rightarrow (4)$$

Write KVL around loop 2:

$$2v_x - v_x + v_L + 1i_x = 0$$
$$v_x + L\frac{di}{dt} + i_x = 0$$
$$4i_x - 2i + 0.1\frac{di}{dt} + i_x = 0$$
$$5i_x - 2i + 0.1\frac{di}{dt} = 0$$

Solving for i_x,

$$i_x = 0.4i - 0.02\frac{di}{dt} \rightarrow (5)$$

Put i_x from (5) in (3),

$$0.01\frac{dv}{dt} + i - (0.4i - 0.02\frac{di}{dt}) = 0$$
$$\frac{dv}{dt} + 60i + 2\frac{di}{dt} = 0 \rightarrow (6)$$

Put i_x from (5) in (4),

$$\frac{di}{dt} - 10v - 40(0.4i - 0.02\frac{di}{dt}) + 20i = 0$$
$$0.18\frac{di}{dt} + 0.4i = v \rightarrow (7)$$

Put v from equation (7) in (6),

$$\frac{d}{dt}[0.18\frac{di}{dt}+0.4i]+60i+2\frac{di}{dt}=0$$
$$0.18\frac{d^2i}{dt^2}+0.4\frac{di}{dt}+60i+2\frac{di}{dt}=0$$
$$\frac{d^2i}{dt^2}+13.33\frac{di}{dt}+333.33i=0 \rightarrow (8)$$

This is the differential equation for current.
Characteristic equation:

$$s^2+13.33s+333.33=0$$

The roots of this equation are:

$$s_{1,2}=-6.67\pm j17$$

From this, $\alpha=6.67$ $\omega_d=17$. so,

$$i_c=e^{-6.67t}[A\cos(17t)+B\sin(17t)]$$

Since the equation is homogeneous and has no forcing function, the particular solution, i_p=0
The total response is:

$$i(t)=i_c+i_p$$
$$i(t)=e^{-6.67t}[A\cos(17t)+B\sin(17t)]$$

When t=0,

$$i(0)=e^0[A\cos(0)+B\sin(0)]$$
$$A=0$$

Now total response is:

$$i(t)=e^{-6.67t}B\sin(17t)$$

Differentiate both sides:

$$\frac{di(t)}{dt}=-6.67e^{-6.67t}B\sin(17t)+17Be^{-6.67t}B\cos(17t)\rightarrow(9)$$

From Equation (7), find $\frac{di(0)}{dt}$:

$$0.18\frac{di(0)}{dt}+0.4i(0)=v(0)$$
$$0.18\frac{di(0)}{dt}+0.4(0)=10$$
$$\frac{di(0)}{dt}=55.56$$

Put this in (9),

$$\frac{di(0)}{dt}=-6.67e^{-6.67(0)}B\sin(17(0))+17Be^{-6.67(0)}B\cos(17(0))$$
$$\frac{di(0)}{dt}=17B$$
$$B=3.27$$

Finally, the equation for the inductor's current is:

$$i(t)=3.27e^{-6.67t}\sin(17t)$$

Now, find out v(t). From equation (7):

$$v(t)=0.18\frac{di}{dt}+0.4i$$

Find di/dt from i(t):

$$\frac{di(t)}{dt}i = 3.27\dot{(} -6.67)e^{-6.67t}\sin(17t) + 3.27\dot{1}7e^{-6.67t}\cos(17t)$$
$$= -21.82e^{-6.67t}\sin(17t) + 55.59e^{-6.67t}\cos(17t)$$

Now solve for v(t):

$$v(t) = 0.18[-21.82e^{-6.67t}\sin(17t) + 55.59e^{-6.67t}\cos(17t)] + 0.4[3.27e^{-6.67t}\sin(17t)]$$
$$v(t) = e^{-6.67t}(10\cos(17t) - 2.62\sin(17t))V$$