LUMS
A Not-for-Profit University

School of Science and Engineering

EE 310 Signals and Systems

PROBLEM SET 3

Preamble (Key Concepts)
This problem set focuses on convolution as the fundamental operation that links an input signal x and an impulse response h to the output y of an LTI system. For continuous-time (CT) signals,

$$y(t) = (x * h)(t) \triangleq \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau,$$

and for discrete-time (DT) signals,

$$y[n] = (x * h)[n] \triangleq \sum_{k=-\infty}^{\infty} x[k]\, h[n-k].$$

A reliable way to compute convolution is to (i) determine when the two signals overlap (support analysis), (ii) set correct limits of integration/summation, and (iii) compute the resulting piecewise expression. You will also use important building blocks and identities:

- Unit step and shifts: $u(t-t_0)$ turns signals on at $t = t_0$.
- Dirac delta and sampling: $\int x(\tau)\delta(\tau - t_0)\, d\tau = x(t_0)$ and $(x * \delta)(t) = x(t)$.
- Time-limited signals: if $x(t)$ is nonzero on $[a, b]$ and $h(t)$ is nonzero on $[c, d]$, then $y(t)$ is nonzero on $[a + c,\, b + d]$.
- Symmetry: even/odd properties can simplify reasoning about the parity of the convolution output.
- Differentiation and convolution (when applicable): $\frac{d}{dt}(x * h) = \left(\frac{dx}{dt}\right) * h = x * \left(\frac{dh}{dt}\right)$.

Several problems also highlight convolution as a smoothing / windowing operation (e.g., rectangular $h(t)$ acts like a finite-duration averager), and reinforce properties such as commutativity, identity, and time-support growth.

Learning Outcomes
By completing this problem set, you should be able to:

1. Compute CT convolution $y(t) = x(t) * h(t)$ using the integral definition, with correct overlap-based limits and clear piecewise results.
2. Compute DT convolution $y[n] = x[n] * h[n]$ using summation, and correctly determine the valid range of n where $y[n] \neq 0$.
3. Apply impulse/step identities (e.g., shifting and scaling with $\delta(\cdot)$ and gating with $u(\cdot)$) to simplify convolution problems quickly and correctly.
4. Use the graphical convolution method (flip–shift–overlap) for time-limited pulses and interpret the resulting shape (triangular/trapezoidal/polynomial pieces).
5. Determine the support (nonzero interval) of a convolution output from the supports of the input signals without evaluating integrals.

6. Interpret convolution with finite-width kernels as filtering/smoothing, and explain how kernel width affects amplitude and waveform smoothness.
7. Relate convolution to differentiation for signals involving derivatives (where conditions allow), and connect slope/discontinuities in $y(t)$ to features of $x(t)$ and $h(t)$.
8. Communicate solutions clearly via sketches, correct notation, and logically structured steps.

Problem 1: Convolution of CT signals

Given

$$x(t) = u(t) - u(t-2), \quad h(t) = u(t),$$

compute the convolution

$$y(t) = x(t) * h(t).$$

Clearly show the limits of integration and sketch all signals.

Problem 2: Convolution with Deltas

Let

$$x(t) = \delta(t-1) + 2\delta(t+1), \quad h(t) = e^{-t}u(t).$$

Find $y(t) = x(t) * h(t)$.

Problem 3: Convolution of rectangular pulses

Given

$$x(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases} \qquad h(t) = \begin{cases} 1, & 0 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$$

evaluate $y(t) = x(t) * h(t)$ using the graphical convolution method.

Problem 4: Convolution of linear segments

Given

$$x(t) = \begin{cases} t, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases} \qquad h(t) = \begin{cases} 1-t, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find $y(t) = x(t) * h(t)$.

Problem 5: Even/Odd signals in convolution

Suppose $x(t)$ is even and $h(t)$ is odd.

(a) Determine whether $y(t) = x(t) * h(t)$ is even, odd, or neither.

(b) Justify your answer mathematically.

Problem 6: DT Convolution

Let

$$x[n] = \begin{cases} 1, & n = 0, 1 \\ 0, & \text{otherwise} \end{cases} \qquad h[n] = \begin{cases} n, & n = 0, 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

Compute $y[n] = x[n] * h[n]$ and specify the range of n.

Problem 7: Non-zero Interval Calculation for Convolution

Let $x(t)$ be nonzero only on $0 \le t \le 2$ and $h(t)$ be nonzero only on $-1 \le t \le 1$.

(a) Determine the interval where $y(t) = x(t) * h(t)$ is nonzero.

(b) Explain your reasoning without computing the integral.

Problem 8: Convolution involving derivatives

Given

$$x(t) = u(t), \quad h(t) = \frac{d}{dt}\big(u(t-1)\big),$$

evaluate $y(t) = x(t) * h(t)$.

Problem 9: Commutativity

Show the following:

(a) Convolution is commutative: $x(t) * h(t) = h(t) * x(t)$.

(b) If both $x(t)$ and $h(t)$ are time-limited, then $y(t)$ is also time-limited.

(c) $x(t) * \delta(t) = x(t)$.

Problem 10: Covolution for exponentials

Let

$$x(t) = e^{-2t}u(t), \quad h(t) = e^{-t}u(t).$$

Compute $y(t) = x(t) * h(t)$ using the standard integral definition of convolution. Sketch the resulting signal.

Problem 11: Covolution for Sinusoids

Let

$$x(t) = \cos(2\pi t)\, u(t), \quad h(t) = u(t) - u(t - 0.5).$$

(a) Compute $y(t) = x(t) * h(t)$ for $0 \le t \le 2$ using integration.

(b) Sketch the first few oscillations of $y(t)$.

(c) Explain how the width of $h(t)$ affects the amplitude and smoothing of $x(t)$.

Problem 12: Covolution involving signum function

Let

$$x(t) = \operatorname{sgn}(t), \quad h(t) = u(t) - u(t-1),$$

where $\operatorname{sgn}(t)$ is the sign function.

(a) Compute $y(t) = x(t) * h(t)$ piecewise.

(b) Discuss the derivative of $y(t)$ and its relation to the original signals.

Problem 13: Covolution for DT involving step functions

Discrete-time convolution:

$$x[n] = \left(-\frac{1}{2}\right)^n u[n], \qquad h[n] = \begin{cases} 1, & 0 \le n \le 7 \\ 0, & \text{otherwise} \end{cases}$$

— End of Problem Set 3 —