LUMS
A Not-for-Profit University

School of Science and Engineering

# EE 310 Signals and Systems

# PROBLEM SET 3 – SOLUTIONS

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## Problem 1: Convolution of CT signals

Given

$$x(t) = u(t) - u(t-2), \quad h(t) = u(t),$$

compute the convolution

$$y(t) = x(t) * h(t).$$

**Solution:** Convolution definition:

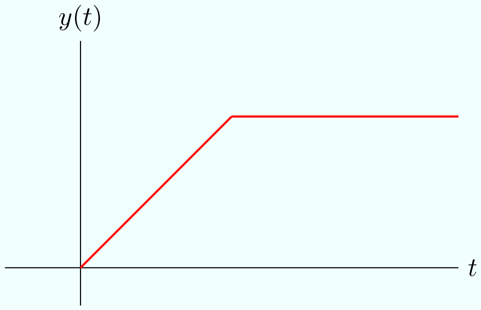
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{0}^{2} 1 \cdot u(t-\tau)d\tau$$

since $x(\tau) = 1$ for $0 \le \tau \le 2$. **Piecewise evaluation:**

- $t < 0$: $y(t) = 0$
- $0 \le t < 2$: $y(t) = \int_0^t 1\, d\tau = t$
- $t \ge 2$: $y(t) = \int_0^2 1\, d\tau = 2$

$$\boxed{y(t) = \begin{cases} 0, & t < 0 \\ t, & 0 \le t < 2 \\ 2, & t \ge 2 \end{cases}}$$

**Graph:**




## Problem 2: Convolution with Deltas

Let

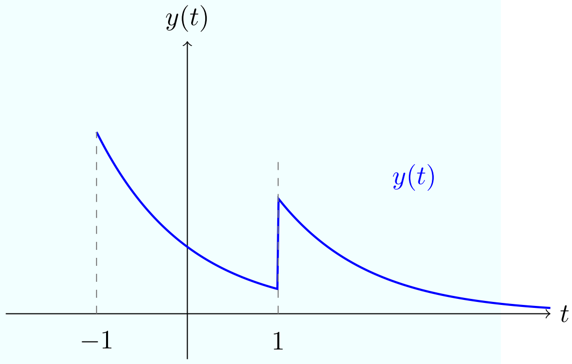
$$x(t) = \delta(t-1) + 2\delta(t+1), \quad h(t) = e^{-t}u(t).$$

**Solution:** Using the property $\delta(t-t_0) * h(t) = h(t-t_0)$:

$$y(t) = h(t-1) + 2h(t+1) = e^{-(t-1)}u(t-1) + 2e^{-(t+1)}u(t+1)$$

$$\boxed{y(t) = e^{-(t-1)}u(t-1) + 2e^{-(t+1)}u(t+1)}$$

**Graph:** Two shifted exponentials starting at $t = -1$ and $t = 1$.




## Problem 3: Convolution of rectangular pulses

Graphical convolution of two rectangular pulses:

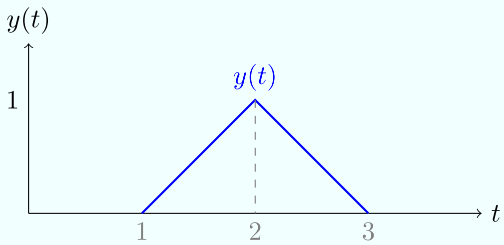
$$x(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}, \quad h(t) = \begin{cases} 1, & 0 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$$

**Solution:** Graphical convolution yields a triangular shape:

- $0 \le t \le 1$: $y(t) = \int_0^t 1 d\tau = 0$
- $1 \le t \le 2$: $y(t) = \int_{t-2}^1 d\tau = t - 1$
- $2 \le t \le 3$: $y(t) = \int_{t-2}^1 d\tau = 3 - t$

$$\boxed{y(t) = \begin{cases} 0, & t < 1 \\ t-1, & 1 \le t < 2 \\ 3-t, & 2 \le t < 3 \\ 0, & t \ge 3 \end{cases}}$$

**Graph:**




## Problem 4: Convolution of linear segments

Convolution of two linear segments:

$$x(t) = t, 0 \le t \le 1; \quad h(t) = 1 - t, 0 \le t \le 1$$

**Solution:**

$$y(t) = \int_0^1 x(\tau)h(t-\tau)d\tau$$

Piecewise evaluation:

**Case 0 $\leq t \leq 1$:**

$$y(t) = \int_0^t \tau(1-(t-\tau))d\tau = \int_0^t \tau(1-t+\tau)d\tau = \frac{t^2}{2} - \frac{t^3}{6} = t^2\left(\frac{3-t}{6}\right)$$

**Case 1 $\leq t \leq 2$:**

$$y(t) = \int_{t-1}^1 \tau(1-(t-\tau))d\tau = \int_{t-1}^1 \tau(1-t+\tau)d\tau = \frac{t^3}{6} - t^2 + \frac{3t}{2} - \frac{2}{3}$$

After computation, the final piecewise:

$$y(t) = \begin{cases} \frac{t^2}{2} - \frac{t^3}{6}, & 0 \leq t \leq 1 \\ \frac{t^3}{6} - t^2 + \frac{3t}{2} - \frac{2}{3}, & 1 \leq t \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

## Problem 5: Even/Odd signals in convolution

Suppose $x(t)$ is even, $h(t)$ is odd.

**Solution:**

1. $y(t) = x(t) * h(t)$ is **odd**.
2. Proof:

$$y(-t) = \int x(\tau)h(-t-\tau)d\tau = \int x(\tau)(-h(t+\tau))d\tau = -y(t)$$

## Problem 6: DT Convolution

Discrete convolution:

$$x[n] = \{1, 1\}, h[n] = \{0, 1, 2\}$$

**Solution:**

$$y[n] = \sum_k x[k]h[n-k]$$

$$y[0] = 1 \cdot 0 = 0, y[1] = 1 \cdot 1 + 1 \cdot 0 = 1, y[2] = 1 \cdot 2 + 1 \cdot 1 = 3, y[3] = 1 \cdot 2 = 2$$

$$n = 0, 1, 2, 3$$

$$\boxed{y[n] = \{0, 1, 3, 2\}, n = 0:3}$$

## Problem 7: Non-zero Interval Calculation for Convolution

Time-limited signals convolution:

$$x : 0 \leq t \leq 2,\ h : -1 \leq t \leq 1$$

**Solution:**

1. Nonzero interval of convolution: $[-1, 3]$.
2. Reason: sum of durations: $t_{\min} = 0 + (-1) = -1$, $t_{\max} = 2 + 1 = 3$.

## Problem 8: Convolution involving derivatives

Let

$$x(t) = u(t), \quad h(t) = \frac{d}{dt}u(t-1)$$

**Solution:** We use the general property of convolution with derivatives:

$$x(t) * \frac{dh(t)}{dt} = \frac{d}{dt}[x(t) * h(t)]$$

Since $h(t) = u(t-1)$, we can write

$$\frac{dh(t)}{dt} = \delta(t-1)$$

Therefore, the convolution is

$$y(t) = x(t) * \delta(t-1)$$

Using the sifting property of the delta function:

$$x(t) * \delta(t - t_0) = x(t - t_0)$$

Hence,

$$\boxed{y(t) = x(t-1) = u(t-1)}$$

**Remark:** This approach works for any $x(t)$. If $h(t)$ is replaced by a general function $g(t)$, then

$$x(t) * \frac{dg(t)}{dt} = \frac{d}{dt}[x(t) * g(t)]$$

which is often easier than directly integrating.

## Problem 9: Commutativity

Properties:

**Solution:**

1. True: $x * h = h * x$ by definition
2. True: convolution duration = sum of durations
3. True: $x * \delta = x$

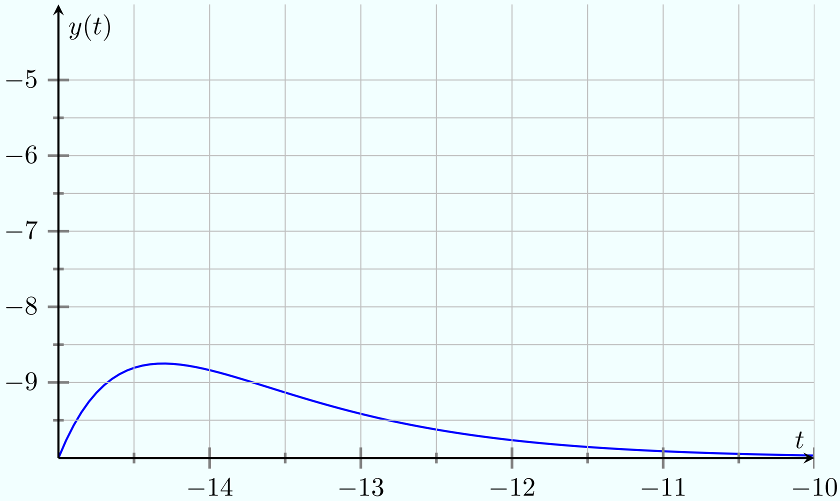
## Problem 10: Covolution for exponentials

Exponential convolution:

$$x = e^{-2t}u(t), h = e^{-t}u(t)$$

**Solution:**

$$y(t) = \int_0^t e^{-2\tau}e^{-(t-\tau)}d\tau = e^{-t}\int_0^t e^{-\tau}d\tau = e^{-t}(1 - e^{-t})$$




**Graph:**

## Problem 11: Covolution for Sinousidals

$$x(t) = \cos(2\pi t)u(t), h(t) = u(t) - u(t - 0.5)$$

**Solution:** We are given

$$x(t) = \sin(2\pi t)\, u(t), \qquad h(t) = u(t) - u(t - 0.5)$$

We are required to find the convolution.

$$y(t) = x(t) * h(t)$$

Convolution is commutative, hence.

$$x(t) * h(t) = h(t) * x(t)$$

Therefore,

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t - \tau)\, d\tau$$

$$y(t) = \int_{-\infty}^{\infty} \big[u(\tau) - u(\tau - 0.5)\big]\, \sin\big(2\pi(t - \tau)\big)\, u(t - \tau)\, d\tau$$

From $h(\tau) = u(\tau) - u(\tau - 0.5)$, we know that

$$h(\tau) \neq 0 \quad \text{only for} \quad 0 \leq \tau < 0.5$$

Also, $u(t - \tau) = 1$ only when

$$\tau \leq t$$

Hence, the practical limits are

$$0 \leq \tau \leq \min(0.5, t)$$

Thus,

$$y(t) = \int_{0}^{\min(0.5,t)} \sin\big(2\pi(t - \tau)\big)\, d\tau$$

Let

$$\alpha = t - \tau \quad \Rightarrow \quad d\alpha = -d\tau$$

When:

$$\tau = 0 \Rightarrow \alpha = t, \qquad \tau = \min(0.5, t) \Rightarrow \alpha = t - \min(0.5, t)$$

Hence,

$$y(t) = \int_{t-\min(0.5,t)}^{t} \sin(2\pi\alpha)\, d\alpha$$

$$\int \sin(2\pi\alpha)\, d\alpha = -\frac{1}{2\pi}\cos(2\pi\alpha)$$

Therefore,

$$y(t) = \frac{1}{2\pi}\Big[\cos\big(2\pi(t - \min(0.5, t))\big) - \cos(2\pi t)\Big]$$

**Case 1:** $t < 0$

No overlap occurs:

$$y(t) = 0$$

**Case 2:** $0 \le t < 0.5$

$$\min(0.5, t) = t$$

$$y(t) = \int_0^t \sin(2\pi(t-\tau))\, d\tau$$

$$\boxed{y(t) = \frac{1}{2\pi}\big[1 - \cos(2\pi t)\big]}$$

**Case 3:** $t \ge 0.5$

$$\min(0.5, t) = 0.5$$

$$y(t) = \int_{t-0.5}^{t} \sin(2\pi\alpha)\, d\alpha$$

Using

$$\cos(2\pi(t-0.5)) = \cos(2\pi t - \pi) = -\cos(2\pi t)$$

$$\boxed{y(t) = -\frac{1}{\pi}\cos(2\pi t)}$$

**Final Answer:**

$$y(t) = \begin{cases} 0, & t < 0, \\ \dfrac{1-\cos(2\pi t)}{2\pi}, & 0 \le t < 0.5, \\ -\dfrac{1}{\pi}\cos(2\pi t), & t \ge 0.5 \end{cases}$$

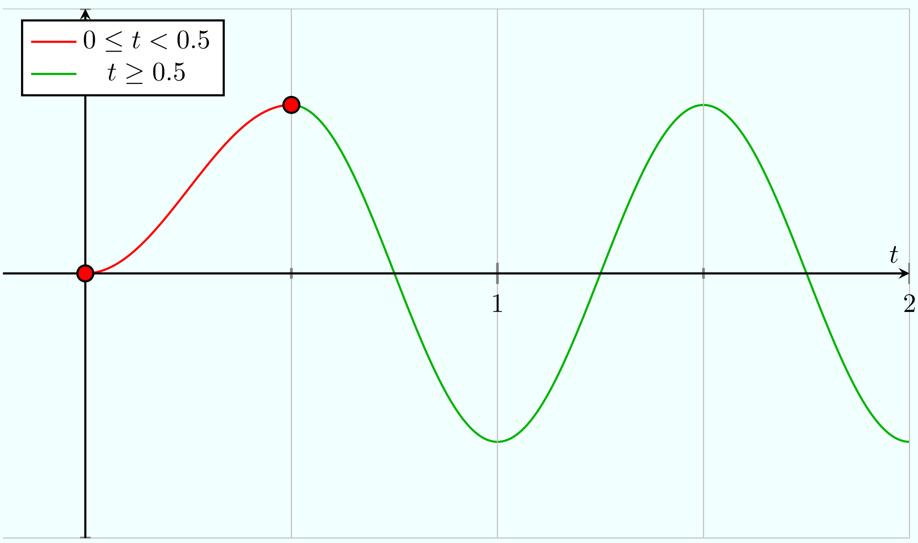



**Figure 1:** Convolution of $\sin(2\pi t)u(t)$ with $u(t) - u(t-0.5)$

## Problem 12: Covolution involving signum function

$$x(t) = \text{sgn}(t), \quad h(t) = u(t) - u(t-1)$$

**Solution:** Convolution definition:

$$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} \text{sgn}(\tau)\, [u(t-\tau) - u(t-\tau-1)]\, d\tau$$

The term $h(t-\tau)$ is nonzero only when

$$0 \le t - \tau \le 1 \implies \tau \in [t-1, t]$$

Hence,

$$y(t) = \int_{t-1}^{t} \text{sgn}(\tau)\, d\tau$$

**Piecewise evaluation:**

1. **Case 1:** $t < 0$
   Interval: $\tau \in [t-1, t] \subset (-\infty, 0)$ So $\text{sgn}(\tau) = -1$:

   $$y(t) = \int_{t-1}^{t} (-1) d\tau = -(t - (t-1)) = -1$$

2. **Case 2:** $0 \le t < 1$
   Interval: $\tau \in [t-1, t]$ crosses zero: $t - 1 < 0 < t$ Split the integral at $\tau = 0$:

   $$y(t) = \int_{t-1}^{0} \text{sgn}(\tau)\, d\tau + \int_{0}^{t} \text{sgn}(\tau)\, d\tau = \int_{t-1}^{0} (-1) d\tau + \int_{0}^{t} 1\, d\tau$$

   Evaluate:

   $$\int_{t-1}^{0} (-1) d\tau = -(0 - (t-1)) = t - 1$$

   $$\int_{0}^{t} 1\, d\tau = t - 0 = t$$

   So total:

   $$y(t) = (t-1) + t = 2t - 1$$

   For $0 \le t < 1$, $y(t) = 2t - 1$.

3. **Case 3:** $1 \le t < 2$
   Interval: $\tau \in [t-1, t] \subset (0, \infty)$ So $\text{sgn}(\tau) = 1$:

   $$y(t) = \int_{t-1}^{t} 1\, d\tau = t - (t-1) = 1$$

4. **Case 4:** $t \ge 2$
   Interval: $\tau \in [t-1, t]$, all positive Also, $h(t-\tau) = 0$ for $\tau > t$ or $\tau < t - 1$ (limit still $[t-1, t]$), $\text{sgn}(\tau) = 1$:

   $$y(t) = \int_{t-1}^{t} 1\, d\tau = 1$$

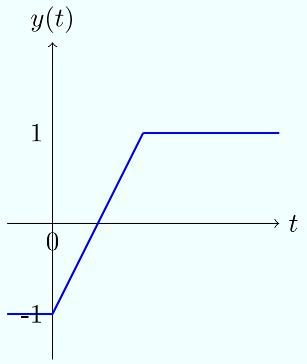
**Final piecewise result:**

$$y(t) = \begin{cases} -1, & t < 0 \\ 2t - 1, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}$$

**Derivative:**

$$\frac{dy}{dt} = \begin{cases} 0, & t < 0 \\ 2, & 0 < t < 1 = 2 * h(t) \\ 0, & t > 1 \end{cases}$$

**Graph:**




## Problem 13: DT Convolution involving step function

Discrete-time convolution:

$$x[n] = \left(-\frac{1}{2}\right)^n u[n], \qquad h[n] = \begin{cases} 1, & 0 \le n \le 7 \\ 0, & \text{otherwise} \end{cases}$$

**Solution:** The discrete-time convolution is defined as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Since $h[n-k] \neq 0$ only when

$$0 \le n-k \le 7 \;\Rightarrow\; n-7 \le k \le n$$

Also, $x[k] \neq 0$ only for $k \ge 0$ due to $u[k]$.
Hence, the practical limits of summation are

$$k = \max(0, n-7) \text{ to } n$$

Therefore,

$$y[n] = \sum_{k=\max(0,n-7)}^{n} \left(-\frac{1}{2}\right)^k$$

This is a finite geometric series.
**Case 1:** $0 \le n \le 7$

$$y[n] = \sum_{k=0}^{n} \left(-\frac{1}{2}\right)^k = \frac{1 - \left(-\frac{1}{2}\right)^{n+1}}{1 + \frac{1}{2}} = \frac{2}{3}\left[1 - \left(-\frac{1}{2}\right)^{n+1}\right]$$

**Case 2:** $n \ge 8$

$$y[n] = \sum_{k=n-7}^{n} \left(-\frac{1}{2}\right)^k = \left(-\frac{1}{2}\right)^{n-7} \sum_{m=0}^{7} \left(-\frac{1}{2}\right)^m$$

$$= \left(-\frac{1}{2}\right)^{n-7} \cdot \frac{1 - \left(-\frac{1}{2}\right)^8}{1 + \frac{1}{2}} = \frac{2}{3}\left(1 - \frac{1}{256}\right)\left(-\frac{1}{2}\right)^{n-7}$$

**Final Answer:**

$$y[n] = \begin{cases} \frac{2}{3}\left[1 - \left(-\frac{1}{2}\right)^{n+1}\right], & 0 \le n \le 7 \\ \frac{2}{3}\left(1 - \frac{1}{256}\right)\left(-\frac{1}{2}\right)^{n-7}, & n \ge 8 \\ 0, & n < 0 \end{cases}$$

— End of Problem Set 3 Solutions —