LUMS
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School of Science and Engineering

# EE 310 Signals and Systems

# PROBLEM SET 4

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## Key Concepts and Equations (Quick Reference)

### 1) Complex Exponentials, Euler Identities, and Eigenfunctions

- **Euler identities:**
$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}, \qquad \sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}.$$
(Same forms with $t \to n$ for discrete time.)
- **Eigenfunction definition:** $x$ is an eigenfunction of a system $\mathcal{T}\{\cdot\}$ if
$$\mathcal{T}\{x\} = \lambda\, x,$$
where $\lambda$ is a (generally complex) constant called the eigenvalue.
- For **CT LTI** systems, $x(t) = e^{st}$ is an eigenfunction and the eigenvalue is
$$\lambda = H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\, d\tau,$$
so $y(t) = H(s)e^{st}$. In particular, for sinusoids ($s = j\omega$), the scaling is $H(j\omega)$.

### 3) Continuous-Time Fourier Series (CTFS)

- **Periodicity:** $x(t)$ is periodic if $x(t + T_0) = x(t)$ for some $T_0 > 0$.
- **Fundamental frequency:** $\omega_0 = \frac{2\pi}{T_0}$.
- **Synthesis Equation:**
$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}.$$
- **Analysis (coefficients):**
$$a_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t)\, e^{-jk\omega_0 t}\, dt \quad (\text{any } t_0).$$
- **Real-signal symmetry:** if $x(t)$ is real, then $a_{-k} = a_k^*$.

**CTFS properties frequently used**

- **Linearity:** $x = x_1 + x_2 \Rightarrow a_k = a_k^{(1)} + a_k^{(2)}$.
- **Time shift:** $y(t) = x(t - t_0) \Rightarrow b_k = a_k e^{-jk\omega_0 t_0}$.
- **Time reversal:** $y(t) = x(-t) \Rightarrow b_k = a_{-k}$.

- **Differentiation:** $y(t) = \frac{d}{dt}x(t) \Rightarrow b_k = jk\omega_0\, a_k$.
- **LTI filtering of periodic inputs:** if $x(t) = \sum a_k e^{jk\omega_0 t}$, then

$$y(t) = \sum b_k e^{jk\omega_0 t}, \qquad b_k = a_k\, H(jk\omega_0).$$

**4) Discrete-Time Fourier Series (DTFS)**

- **Periodicity:** $x[n]$ is periodic if $x[n+N] = x[n]$ for some *integer* $N > 0$.
- **Synthesis:**

$$x[n] = \sum_{k=0}^{N-1} a_k\, e^{j\frac{2\pi}{N}kn}.$$

- **Analysis:**

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}, \qquad k = 0, 1, \ldots, N-1.$$

- **Periodicity in index:** $a_{k+N} = a_k$.
- **Real-signal symmetry:** if $x[n]$ is real, then (note the mod, that is because of periodicity in index)

$$a_k = a^*_{(-k \bmod N)}.$$

**DTFS properties frequently used**

- **Time shift:** $y[n] = x[n-n_0] \Rightarrow b_k = a_k e^{-j\frac{2\pi}{N}kn_0}$.
- **LTI filtering of periodic inputs:** if $x[n] = \sum a_k e^{j\frac{2\pi}{N}kn}$, then

$$y[n] = \sum b_k e^{j\frac{2\pi}{N}kn}, \qquad b_k = a_k\, H\left(e^{j\frac{2\pi}{N}k}\right).$$

**5) Parseval's Relation (Average Power)**

- **CT periodic:**

$$P = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |a_k|^2.$$

- **DT periodic:**

$$P = \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |a_k|^2.$$

**6) Useful Series Sums (often needed in DTFS, requested by the students**

- Finite geometric sum ($r \neq 1$):

$$\sum_{n=0}^{M} r^n = \frac{1 - r^{M+1}}{1-r}.$$

- Infinite geometric sum ($|r| < 1$):

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

## Problem 1

Complex exponentials are fundamental to the study of Linear Time-Invariant (LTI) systems because they act as eigenfunctions.

(a) Define what it means for a signal to be an eigenfunction of a system.

(b) Prove that the complex exponential $x(t) = e^{st}$ is an eigenfunction of a continuous-time LTI system with impulse response $h(t)$.

(c) If an LTI system has an impulse response $h(t) = e^{-3t}u(t)$, find the corresponding eigenvalue when the input is $x(t) = e^{j2t}$.

## Problem 2

Consider the continuous-time signal $x(t) = 4\cos(10\pi t) - 2\sin(15\pi t)$.

(a) Determine the fundamental period $T_0$ and fundamental frequency $\omega_0$ of $x(t)$.

(b) Express $x(t)$ as a Continuous-Time Fourier Series (CTFS) using complex exponentials.

(c) Identify the non-zero Fourier coefficients $a_k$.

## Problem 3

Let $x(t)$ be a periodic square wave with period $T = 4$. Over one period, the signal is defined as:

$$x(t) = \begin{cases} 1, & |t| < 1 \\ 0, & 1 \le |t| \le 2 \end{cases}$$

(a) Compute the DC component (average value) $a_0$ of the signal.

(b) Determine the general expression for the CTFS coefficients $a_k$ for $k \neq 0$.

(c) Sketch the magnitude spectrum $|a_k|$ for $k \in [-3, 3]$.

## Problem 4

Suppose a periodic signal $x(t)$ has fundamental period $T = 2$ and Fourier coefficients $a_k$. Use the properties of the CTFS to find the Fourier coefficients of the following signals in terms of $a_k$:

(a) $y_1(t) = x(t - 0.5)$

(b) $y_2(t) = \frac{dx(t)}{dt}$

(c) $y_3(t) = x(-t)$

## Problem 5

A continuous-time LTI system has the frequency response $H(j\omega) = \frac{1}{1+j\omega/2}$. The input to the system is $x(t) = 2 + 4\cos(2t)$.

(a) Find the CTFS coefficients of the input signal $x(t)$.

(b) Calculate the corresponding CTFS coefficients of the output signal $y(t)$.

(c) Write the output signal $y(t)$ in the time domain as a sum of real functions.

## Problem 6

Consider the discrete-time signal $x[n] = 2 + 2\sin\left(\frac{2\pi}{5}n + \frac{\pi}{4}\right)$.

(a) What is the fundamental period $N$ of this sequence?

(b) Express $x[n]$ in terms of complex exponentials.

(c) Determine the Discrete-Time Fourier Series (DTFS) coefficients $a_k$ for $0 \le k \le N-1$.

## Problem 7

Let $x[n]$ be a periodic discrete-time signal with period $N = 8$, defined over one period $0 \le n \le 7$ as:

$$x[n] = \begin{cases} 1, & 0 \le n \le 3 \\ 0, & 4 \le n \le 7 \end{cases}$$

(a) Calculate the DC coefficient $a_0$.

(b) Evaluate the DTFS coefficients $a_k$ using the standard summation formula.

(c) Show that $a_{k+8} = a_k$ using your derived expression.

## Problem 8

A discrete-time periodic signal $x[n]$ has period $N = 4$ and DTFS coefficients $a_0 = 1$, $a_1 = j$, $a_2 = 2$, $a_3 = -j$.

(a) Find the DTFS coefficients of the time-shifted signal $y[n] = x[n-1]$.

(b) Determine whether the original time-domain signal $x[n]$ is completely real. Justify your answer.

(c) Calculate the average power of $x[n]$ using Parseval's relation.

## Problem 9

A discrete-time LTI system is described by the impulse response $h[n] = \left(\frac{1}{2}\right)^n u[n]$.

(a) Find the frequency response $H(e^{j\omega})$ of this system.

(b) The input to the system is a periodic sequence $x[n] = (-1)^n$. Find the fundamental period $N$ and the DTFS coefficients $a_k$ of $x[n]$.

(c) Compute the steady-state output $y[n]$ of the system.

## Problem 10

Consider a basic series RC circuit where the input $x(t)$ is the source voltage and the output $y(t)$ is the voltage across the capacitor. The linear constant-coefficient differential equation relating $x(t)$ and $y(t)$ is given by

$$x(t) = v_R(t) + v_C(t) = RC\frac{dy(t)}{dt} + y(t)$$

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(a) Given $RC = 1$ second, find the impulse response $h(t)$ of the system.

(b) Is this system memoryless? Is it causal? Is it stable? Justify briefly.

— End of Problem Set —