LUMS
A Not-for-Profit University

School of Science and Engineering

EE 310 Signals and Systems

PROBLEM SET 4

Key Concepts and Equations (Quick Reference)

1) Complex Exponentials, Euler Identities, and Eigenfunctions

- **Euler identities:**

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}, \qquad \sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}.$$

 (Same forms with $t \to n$ for discrete time.)

- **Eigenfunction definition:** x is an eigenfunction of a system $\mathcal{T}\{\cdot\}$ if

$$\mathcal{T}\{x\} = \lambda\, x,$$

 where λ is a (generally complex) constant called the eigenvalue.

- For **CT LTI** systems, $x(t) = e^{st}$ is an eigenfunction and the eigenvalue is

$$\lambda = H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau}\, d\tau,$$

 so $y(t) = H(s)e^{st}$. In particular, for sinusoids ($s = j\omega$), the scaling is $H(j\omega)$.

3) Continuous-Time Fourier Series (CTFS)

- **Periodicity:** $x(t)$ is periodic if $x(t + T_0) = x(t)$ for some $T_0 > 0$.
- **Fundamental frequency:** $\omega_0 = \frac{2\pi}{T_0}$.
- **Synthesis Equation:**

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}.$$

- **Analysis (coefficients):**

$$a_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t)\, e^{-jk\omega_0 t}\, dt \quad \text{(any } t_0\text{)}.$$

- **Real-signal symmetry:** if $x(t)$ is real, then $a_{-k} = a_k^*$.

CTFS properties frequently used

- **Linearity:** $x = x_1 + x_2 \Rightarrow a_k = a_k^{(1)} + a_k^{(2)}$.
- **Time shift:** $y(t) = x(t - t_0) \Rightarrow b_k = a_k e^{-jk\omega_0 t_0}$.
- **Time reversal:** $y(t) = x(-t) \Rightarrow b_k = a_{-k}$.

- **Differentiation:** $y(t) = \frac{d}{dt}x(t) \Rightarrow b_k = jk\omega_0\, a_k$.
- **LTI filtering of periodic inputs:** if $x(t) = \sum a_k e^{jk\omega_0 t}$, then

$$y(t) = \sum b_k e^{jk\omega_0 t}, \qquad b_k = a_k\, H(jk\omega_0).$$

4) Discrete-Time Fourier Series (DTFS)

- **Periodicity:** $x[n]$ is periodic if $x[n+N] = x[n]$ for some *integer* $N > 0$.
- **Synthesis:**

$$x[n] = \sum_{k=0}^{N-1} a_k\, e^{j\frac{2\pi}{N}kn}.$$

- **Analysis:**

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}, \qquad k = 0, 1, \ldots, N-1.$$

- **Periodicity in index:** $a_{k+N} = a_k$.
- **Real-signal symmetry:** if $x[n]$ is real, then (note the mod, that is because of periodicity in index)

$$a_k = a^*_{(-k \bmod N)}.$$

DTFS properties frequently used

- **Time shift:** $y[n] = x[n - n_0] \Rightarrow b_k = a_k e^{-j\frac{2\pi}{N}kn_0}$.
- **LTI filtering of periodic inputs:** if $x[n] = \sum a_k e^{j\frac{2\pi}{N}kn}$, then

$$y[n] = \sum b_k e^{j\frac{2\pi}{N}kn}, \qquad b_k = a_k\, H\left(e^{j\frac{2\pi}{N}k}\right).$$

5) Parseval's Relation (Average Power)

- **CT periodic:**

$$P = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |a_k|^2.$$

- **DT periodic:**

$$P = \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |a_k|^2.$$

6) Useful Series Sums (often needed in DTFS, requested by the students

- Finite geometric sum ($r \neq 1$):

$$\sum_{n=0}^{M} r^n = \frac{1 - r^{M+1}}{1 - r}.$$

- Infinite geometric sum ($|r| < 1$):

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}.$$

Problem 1

Complex exponentials are fundamental to the study of Linear Time-Invariant (LTI) systems because they act as eigenfunctions.

(a) Define what it means for a signal to be an eigenfunction of a system.

(b) Prove that the complex exponential $x(t) = e^{st}$ is an eigenfunction of a continuous-time LTI system with impulse response $h(t)$.

(c) If an LTI system has an impulse response $h(t) = e^{-3t}u(t)$, find the corresponding eigenvalue when the input is $x(t) = e^{j2t}$.

Solution:

(a) An eigenfunction is a signal that passes through a system and emerges perfectly intact, scaled only by a complex constant (the eigenvalue).

(b)

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

Since $\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau = H(s)$ is a constant with respect to t, $y(t) = H(s)e^{st}$.

(c)

$$H(j2) = \int_{0}^{\infty} e^{-3\tau}e^{-j2\tau}d\tau = \frac{1}{3+j2}$$

$$\boxed{\text{Eigenvalue: } \frac{1}{3+j2}}$$

Problem 2

Consider the continuous-time signal $x(t) = 4\cos(10\pi t) - 2\sin(15\pi t)$.

(a) Determine the fundamental period T_0 and fundamental frequency ω_0 of $x(t)$.

(b) Express $x(t)$ as a Continuous-Time Fourier Series (CTFS) using complex exponentials.

(c) Identify the non-zero Fourier coefficients a_k.

Solution:

(a) $\omega_1 = 10\pi$, $\omega_2 = 15\pi$. The greatest common divisor is $\omega_0 = 5\pi$.

$$T_0 = \frac{2\pi}{\omega_0} = \frac{2}{5} \text{ seconds}$$

(b) Using Euler's identity:

$$x(t) = 2(e^{j10\pi t} + e^{-j10\pi t}) - \frac{1}{j}(e^{j15\pi t} - e^{-j15\pi t})$$

(c) Since $\omega_0 = 5\pi$, 10π is $k = 2$ and 15π is $k = 3$.

$$\boxed{a_2 = 2, \quad a_{-2} = 2, \quad a_3 = j, \quad a_{-3} = -j}$$

All other $a_k = 0$.

Problem 3

Let $x(t)$ be a periodic square wave with period $T = 4$. Over one period, the signal is defined as:

$$x(t) = \begin{cases} 1, & |t| < 1 \\ 0, & 1 \leq |t| \leq 2 \end{cases}$$

(a) Compute the DC component (average value) a_0 of the signal.

(b) Determine the general expression for the CTFS coefficients a_k for $k \neq 0$.

(c) Sketch the magnitude spectrum $|a_k|$ for $k \in [-3, 3]$.

Solution:

(a)

$$a_0 = \frac{1}{4}\int_{-1}^{1} 1\, dt = \frac{1}{2}$$

(b) $\omega_0 = 2\pi/4 = \pi/2$.

$$a_k = \frac{1}{4}\int_{-1}^{1} e^{-jk(\pi/2)t}\, dt = \frac{1}{4}\left[\frac{e^{-jk(\pi/2)t}}{-jk\pi/2}\right]_{-1}^{1} = \frac{\sin(k\pi/2)}{k\pi}$$

(c) $|a_0| = 0.5$, $|a_1| = |a_{-1}| = 1/\pi$, $a_2 = a_{-2} = 0$, $|a_3| = |a_{-3}| = 1/(3\pi)$. (Sketch accordingly).

Problem 4

Suppose a periodic signal $x(t)$ has fundamental period $T = 2$ and Fourier coefficients a_k. Use the properties of the CTFS to find the Fourier coefficients of the following signals in terms of a_k:

(a) $y_1(t) = x(t - 0.5)$

(b) $y_2(t) = \frac{dx(t)}{dt}$

(c) $y_3(t) = x(-t)$

Solution:

(a) Time-shifting property: $b_k = a_k e^{-jk\omega_0 t_0}$. Here $\omega_0 = \pi, t_0 = 0.5$.

$$\boxed{b_k = a_k e^{-jk\pi/2}}$$

(b) Differentiation property:

$$\boxed{c_k = jk\pi a_k}$$

(c) Time-reversal property:

$$\boxed{d_k = a_{-k}}$$

Problem 5

A continuous-time LTI system has the frequency response $H(j\omega) = \frac{1}{1+j\omega/2}$. The input to the system is $x(t) = 2 + 4\cos(2t)$.

(a) Find the CTFS coefficients of the input signal $x(t)$.

(b) Calculate the corresponding CTFS coefficients of the output signal $y(t)$.

(c) Write the output signal $y(t)$ in the time domain as a sum of real functions.

Solution:

(a) $\omega_0 = 2$. $x(t) = 2 + 2e^{j2t} + 2e^{-j2t}$.

$$a_0 = 2, \quad a_1 = 2, \quad a_{-1} = 2$$

(b) $b_k = a_k H(jk\omega_0) = a_k H(j2k)$. $b_0 = 2 \cdot H(0) = 2 \cdot 1 = 2$. $b_1 = 2 \cdot H(j2) = 2 \cdot \frac{1}{1+j} = \frac{2}{\sqrt{2}} e^{-j\pi/4} = \sqrt{2}e^{-j\pi/4}$. $b_{-1} = \sqrt{2}e^{j\pi/4}$.

(c)

$$\boxed{y(t) = 2 + 2\sqrt{2}\cos(2t - \pi/4)}$$

Problem 6

Consider the discrete-time signal $x[n] = 2 + 2\sin\left(\frac{2\pi}{5}n + \frac{\pi}{4}\right)$.

(a) What is the fundamental period N of this sequence?

(b) Express $x[n]$ in terms of complex exponentials.

(c) Determine the Discrete-Time Fourier Series (DTFS) coefficients a_k for $0 \leq k \leq N-1$.

Solution:

(a) The fundamental period is strictly an integer.

$$\boxed{N = 5}$$

(b)

$$x[n] = 2 + \frac{1}{j}e^{j(2\pi/5)n}e^{j\pi/4} - \frac{1}{j}e^{-j(2\pi/5)n}e^{-j\pi/4}$$

(c) For $N = 5$, $k = -1$ corresponds to $k = 4$. $a_0 = 2$. $a_1 = \frac{1}{j}e^{j\pi/4} = e^{-j\pi/2}e^{j\pi/4} = e^{-j\pi/4}$. $a_4 = a_{-1} = -\frac{1}{j}e^{-j\pi/4} = e^{j\pi/4}$. $a_2 = a_3 = 0$.

Problem 7

Let $x[n]$ be a periodic discrete-time signal with period $N = 8$, defined over one period $0 \leq n \leq 7$ as:

$$x[n] = \begin{cases} 1, & 0 \leq n \leq 3 \\ 0, & 4 \leq n \leq 7 \end{cases}$$

(a) Calculate the DC coefficient a_0.

(b) Evaluate the DTFS coefficients a_k using the standard summation formula.

(c) Show that $a_{k+8} = a_k$ using your derived expression.

Solution:

(a)

$$a_0 = \frac{1}{8}\sum_{n=0}^{7} x[n] = \frac{4}{8} = \frac{1}{2}$$

(b) $\omega_0 = 2\pi/8 = \pi/4$. Using the geometric series sum formula for $k \neq 0$:

$$a_k = \frac{1}{8}\sum_{n=0}^{3} e^{-jk(\pi/4)n} = \frac{1}{8}\frac{1 - e^{-jk\pi}}{1 - e^{-jk\pi/4}}$$

(c) Substituting $k + 8$ into the exponential: $e^{-j(k+8)\pi/4} = e^{-jk\pi/4}e^{-j2\pi} = e^{-jk\pi/4}$. Thus the ratio remains completely unchanged, proving $a_{k+8} = a_k$.

Problem 8

A discrete-time periodic signal $x[n]$ has period $N = 4$ and DTFS coefficients $a_0 = 1$, $a_1 = j$, $a_2 = 2$, $a_3 = -j$.

(a) Find the DTFS coefficients of the time-shifted signal $y[n] = x[n-1]$.

(b) Determine whether the original time-domain signal $x[n]$ is completely real. Justify your answer.

(c) Calculate the average power of $x[n]$ using Parseval's relation.

Solution:

(a) $b_k = a_k e^{-jk(2\pi/4)1} = a_k e^{-jk\pi/2}$. $b_0 = 1$, $b_1 = j(-j) = 1$, $b_2 = 2(-1) = -2$, $b_3 = -j(j) = 1$.

(b) For $x[n]$ to be real, $a_k = a^*_{-k}$. In DTFS, $a_{-1} = a_{N-1} = a_3$. Checking: $a_1 = j$. $a_3^* = (-j)^* = j$. Since $a_1 = a_3^*$ and a_0, a_2 are real, $x[n]$ is real.

(c) Parseval's relation: $P = \sum_{k=0}^{3} |a_k|^2$.

$$P = |1|^2 + |j|^2 + |2|^2 + |-j|^2 = 1 + 1 + 4 + 1 = \boxed{7}$$

Problem 9

A discrete-time LTI system is described by the impulse response $h[n] = \left(\frac{1}{2}\right)^n u[n]$.

(a) Find the frequency response $H(e^{j\omega})$ of this system.

(b) The input to the system is a periodic sequence $x[n] = (-1)^n$. Find the fundamental period N and the DTFS coefficients a_k of $x[n]$.

(c) Compute the steady-state output $y[n]$ of the system.

Solution:

(a)

$$H(e^{j\omega}) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\omega n} = \frac{1}{1 - 0.5e^{-j\omega}}$$

(b) $x[n] = (-1)^n = e^{j\pi n}$. The fundamental frequency is π, so $N = 2$. $a_1 = 1$, $a_0 = 0$.

(c) The input only contains the frequency $\omega = \pi$. $H(e^{j\pi}) = \frac{1}{1-0.5(-1)} = \frac{1}{1.5} = \frac{2}{3}$.

$$\boxed{y[n] = \frac{2}{3}(-1)^n}$$

Problem 10

Consider a basic series RC circuit where the input $x(t)$ is the source voltage and the output $y(t)$ is the voltage across the capacitor. The linear constant-coefficient differential equation relating $x(t)$ and $y(t)$ is given by

$$x(t) = v_R(t) + v_C(t) = RC\frac{dy(t)}{dt} + y(t)$$

.

(a) Given $RC = 1$ second, find the impulse response $h(t)$ of the system.

(b) Is this system memoryless? Is it causal? Is it stable? Justify briefly.

Solution:

(a) $RC = 1 \Rightarrow \frac{dy(t)}{dt} + y(t) = x(t)$. Solving for the impulse response yields:

$$\boxed{h(t) = e^{-t}u(t)}$$

(b) - **Not memoryless**: $h(t)$ is not a scaled impulse.
- **Causal**: $h(t) = 0$ for $t < 0$.
- **Stable**: $\int_{-\infty}^{\infty} |h(t)|dt = \int_0^{\infty} e^{-t}dt = 1 < \infty$.

— End of Problem Set —