

Key Concepts and Equations (Quick Reference)

While the Fourier Series allows us to represent periodic signals as a sum of harmonically related complex exponentials, the Fourier Transform extends this powerful concept to **aperiodic signals**. By letting the period of a signal approach infinity, the discrete frequency spectrum of the Fourier Series evolves into a continuous frequency spectrum. Before diving into the problems, please review the following core concepts and properties. Let $x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$ and $y(t) \xleftrightarrow{\mathcal{F}} Y(\omega)$.

1. The CTFT Pair

- **Analysis Equation (Forward Transform):**

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

- **Synthesis Equation (Inverse Transform):**

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega$$

2. Symmetry Properties

For any strictly real-valued signal $x(t)$, its Fourier transform exhibits conjugate symmetry, meaning $X(\omega) = X^*(-\omega)$.

- If $x(t)$ is real and **even**, $X(\omega)$ is purely real and even.
- If $x(t)$ is real and **odd**, $X(\omega)$ is purely imaginary and odd.

3. Properties of the CTFT

These operational properties allow you to find transforms without computing complex integrals from scratch:

- **Linearity:** $ax(t) + by(t) \xleftrightarrow{\mathcal{F}} aX(\omega) + bY(\omega)$
- **Time Shifting:** $x(t - t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0} X(\omega)$
- **Frequency Shifting:** $e^{j\omega_0 t} x(t) \xleftrightarrow{\mathcal{F}} X(\omega - \omega_0)$
- **Time Scaling:** $x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$
- **Conjugation:** $x^*(t) \xleftrightarrow{\mathcal{F}} X^*(-\omega)$
- **Duality:** $X(t) \xleftrightarrow{\mathcal{F}} 2\pi x(-\omega)$
- **Differentiation in Time:** $\frac{dx(t)}{dt} \xleftrightarrow{\mathcal{F}} j\omega X(\omega)$

- **Differentiation in Frequency:** $tx(t) \xleftrightarrow{\mathcal{F}} j \frac{d}{d\omega} X(\omega)$
- **Integration:** $\int_{-\infty}^t x(\tau) d\tau \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega} X(\omega) + \pi X(0) \delta(\omega)$
- **Convolution (Time):** $x(t) * y(t) \xleftrightarrow{\mathcal{F}} X(\omega) Y(\omega)$
- **Multiplication (Time):** $x(t)y(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} [X(\omega) * Y(\omega)]$
- **Parseval's Relation:** $\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$

4. LTI Systems and Frequency Response

For a causal Linear Time-Invariant (LTI) system, the frequency response $H(\omega)$ completely characterizes the system's behavior. It is defined as the ratio of the output spectrum to the input spectrum:

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

For systems described by Linear Constant-Coefficient Differential Equations (LCCDEs), you can find $H(\omega)$ by applying the Fourier transform to the entire differential equation, converting calculus into algebra.

5. CTFT of Periodic Signals

Although originally derived for aperiodic signals, the CTFT can represent periodic signals using the Dirac delta function $\delta(\omega)$. If a periodic signal has Fourier series coefficients a_k and a fundamental frequency ω_0 , its Fourier transform is a train of impulses situated at harmonic frequencies:

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

Problem 1

Consider the signal $x(t)$, which consists of a single rectangular pulse of unit height, is symmetric about the origin, and has a total width T_1 .

- Sketch $x(t)$.
- Sketch $\tilde{x}(t)$, which is a periodic repetition of $x(t)$ with period $T_0 = 3T_1/2$.
- Compute $X(\omega)$, the Fourier transform of $x(t)$. Sketch $|X(\omega)|$ for $|\omega| \leq 6\pi/T_1$.
- Compute a_k , the Fourier series coefficients of $\tilde{x}(t)$. Sketch a_k for $k = 0, \pm 1, \pm 2, \pm 3$.
- Using your answers to (c) and (d), verify that, for this example, $a_k = \frac{1}{T_0} X(\omega) \Big|_{\omega=k\omega_0}$.
- Write a statement that indicates how the Fourier series for a periodic function can be obtained if the Fourier transform of one period of this periodic function is given.

Problem 2

Determine the Fourier transform of $x(t) = e^{-t/2}u(t)$ and sketch:

- $|X(\omega)|$
- $\angle X(\omega)$
- $\text{Re}\{X(\omega)\}$
- $\text{Im}\{X(\omega)\}$

Problem 3

By considering the Fourier analysis equation or synthesis equation, show the validity in general of each of the following statements:

- If $x(t)$ is real-valued, then $X(\omega) = X^*(-\omega)$.
 - If $x(t) = x^*(-t)$, then $X(\omega)$ is real-valued.
- (b) Using the statements in part (a), show the validity of each of the following statements:
- If $x(t)$ is real and even, then $X(\omega)$ is real and even.
 - If $x(t)$ is real and odd, then $X(\omega)$ is imaginary and odd.

Problem 4

The output of a causal LTI system is related to the input $x(t)$ by the differential equation:

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

- Determine the frequency response $H(\omega) = Y(\omega)/X(\omega)$ and sketch the phase and magnitude of $H(\omega)$.
- If $x(t) = e^{-t}u(t)$, determine $Y(\omega)$, the Fourier transform of the output.
- Find $y(t)$ for the input given in part (b).

Problem 5

Compute the Fourier transform of each of the following signals:

- $[e^{-at} \cos \omega_0 t]u(t)$, $a > 0$
- $e^{-3|t|} \sin 2t$
- $\frac{\sin \pi t}{\pi t} \cdot \frac{\sin 2\pi t}{\pi t}$

Problem 6

Use properties of the Fourier transform to show by induction that the Fourier transform of

$$x(t) = \frac{t^{n-1}}{(n-1)!} e^{-at} u(t), \quad a > 0$$

is

$$X(\omega) = \frac{1}{(a + j\omega)^n}$$

Problem 7

In this problem, we explore the definition of the Fourier transform of a periodic signal.

- (a) Show that if $x_3(t) = ax_1(t) + bx_2(t)$, then $X_3(\omega) = aX_1(\omega) + bX_2(\omega)$.
(b) Verify that

$$e^{j\omega_0 t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t} d\omega$$

From this observation, argue that the Fourier transform of $e^{j\omega_0 t}$ is $2\pi\delta(\omega - \omega_0)$.

- (c) Recall the synthesis equation for the Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

By taking the Fourier transform of both sides and using the results to parts (a) and (b), show that

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

— End of Problem Set —