LUMS
A Not-for-Profit University

School of Science and Engineering

EE 310 Signals and Systems

PROBLEM SET 5 – SOLUTIONS

Key Concepts and Equations (Quick Reference)

While the Fourier Series allows us to represent periodic signals as a sum of harmonically related complex exponentials, the Fourier Transform extends this powerful concept to **aperiodic signals**. By letting the period of a signal approach infinity, the discrete frequency spectrum of the Fourier Series evolves into a continuous frequency spectrum. Before diving into the problems, please review the following core concepts and properties. Let $x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega)$ and $y(t) \stackrel{\mathcal{F}}{\longleftrightarrow} Y(\omega)$.

1. The CTFT Pair

- **Analysis Equation (Forward Transform):**

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

- **Synthesis Equation (Inverse Transform):**

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

2. Symmetry Properties
For any strictly real-valued signal $x(t)$, its Fourier transform exhibits conjugate symmetry, meaning $X(\omega) = X^*(-\omega)$.

- If $x(t)$ is real and **even**, $X(\omega)$ is purely real and even.
- If $x(t)$ is real and **odd**, $X(\omega)$ is purely imaginary and odd.

3. Properties of the CTFT
These operational properties allow you to find transforms without computing complex integrals from scratch:

- **Linearity:** $ax(t) + by(t) \stackrel{\mathcal{F}}{\longleftrightarrow} aX(\omega) + bY(\omega)$
- **Time Shifting:** $x(t - t_0) \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0}X(\omega)$
- **Frequency Shifting:** $e^{j\omega_0 t}x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega - \omega_0)$
- **Time Scaling:** $x(at) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$
- **Conjugation:** $x^*(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X^*(-\omega)$
- **Duality:** $X(t) \stackrel{\mathcal{F}}{\longleftrightarrow} 2\pi x(-\omega)$
- **Differentiation in Time:** $\frac{dx(t)}{dt} \stackrel{\mathcal{F}}{\longleftrightarrow} j\omega X(\omega)$

- **Differentiation in Frequency:** $tx(t) \overset{\mathcal{F}}{\longleftrightarrow} j\frac{d}{d\omega}X(\omega)$
- **Integration:** $\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
- **Convolution (Time):** $x(t) * y(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)Y(\omega)$
- **Multiplication (Time):** $x(t)y(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}[X(\omega) * Y(\omega)]$
- **Parseval's Relation:** $\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$

4. LTI Systems and Frequency Response

For a causal Linear Time-Invariant (LTI) system, the frequency response $H(\omega)$ completely characterizes the system's behavior. It is defined as the ratio of the output spectrum to the input spectrum:

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

For systems described by Linear Constant-Coefficient Differential Equations (LCCDEs), you can find $H(\omega)$ by applying the Fourier transform to the entire differential equation, converting calculus into algebra.

5. CTFT of Periodic Signals

Although originally derived for aperiodic signals, the CTFT can represent periodic signals using the Dirac delta function $\delta(\omega)$. If a periodic signal has Fourier series coefficients a_k and a fundamental frequency ω_0, its Fourier transform is a train of impulses situated at harmonic frequencies:

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

Problem 1

Consider the signal $x(t)$, which consists of a single rectangular pulse of unit height, is symmetric about the origin, and has a total width T_1.

(a) Sketch $x(t)$.

(b) Sketch $\tilde{x}(t)$, which is a periodic repetition of $x(t)$ with period $T_0 = 3T_1/2$.

(c) Compute $X(\omega)$, the Fourier transform of $x(t)$. Sketch $|X(\omega)|$ for $|\omega| \leq 6\pi/T_1$.

(d) Compute a_k, the Fourier series coefficients of $\tilde{x}(t)$. Sketch a_k for $k = 0, \pm1, \pm2, \pm3$.

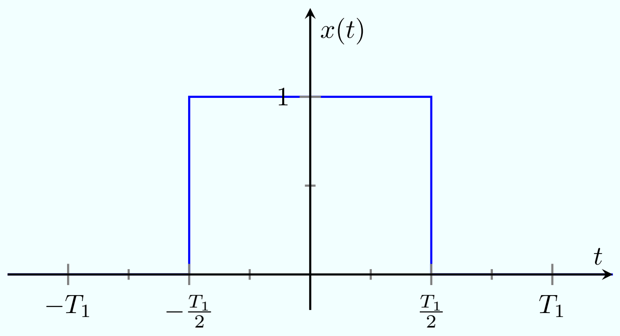
(e) Using your answers to (c) and (d), verify that, for this example, $a_k = \frac{1}{T_0} X(\omega)\big|_{\omega=k\omega_0}$.

(f) Write a statement that indicates how the Fourier series for a periodic function can be obtained if the Fourier transform of one period of this periodic function is given.

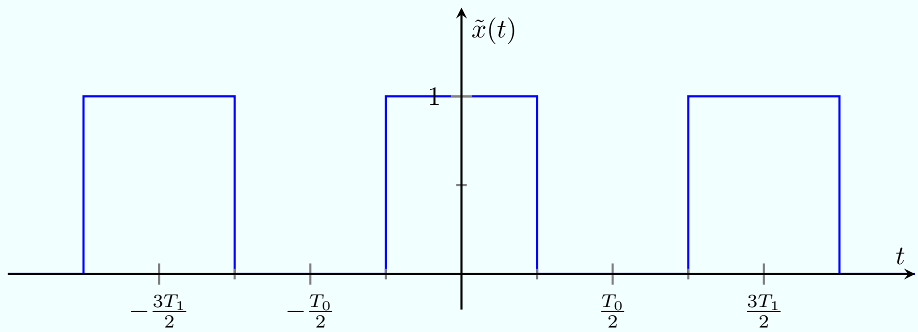
Solution:

(a) $x(t)$ is a rectangular pulse symmetric about origin:

$$x(t) = \begin{cases} 1, & |t| \leq T_1/2 \\ 0, & |t| > T_1/2 \end{cases}$$



(b) $\tilde{x}(t)$ is the periodic extension of $x(t)$ with period T_0.



(c) Using the definition of the Fourier transform,

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Since $x(t) = 0$ for $|t| > T_1/2$,

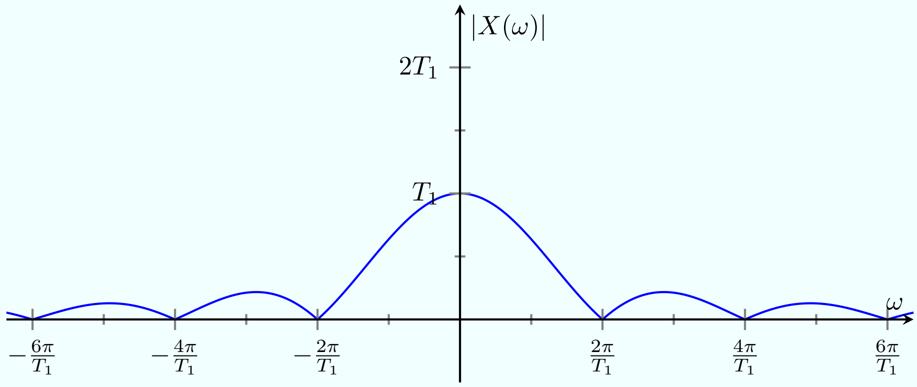
$$X(\omega) = \int_{-T_1/2}^{T_1/2} e^{-j\omega t} dt$$

$$X(\omega) = \left[\frac{e^{-j\omega t}}{-j\omega} \right]_{-T_1/2}^{T_1/2}$$

$$X(\omega) = \frac{-1}{j\omega} \left(e^{-j\omega T_1/2} - e^{j\omega T_1/2} \right)$$

Using $e^{j\theta} - e^{-j\theta} = 2j\sin\theta$,

$$X(\omega) = \frac{2\sin(\omega T_1/2)}{\omega}$$



(d) Using the Fourier series analysis formula,

$$a_k = \frac{1}{T_0}\int_{T_0} \tilde{x}(t)e^{-jk\omega_0 t}dt$$

Since $\tilde{x}(t) = x(t)$ within one pulse,

$$a_k = \frac{1}{T_0}\int_{-T_1/2}^{T_1/2} e^{-jk\omega_0 t}dt$$

$$a_k = \frac{1}{T_0}\frac{-1}{jk\omega_0}\left(e^{-jk\omega_0 T_1/2} - e^{jk\omega_0 T_1/2}\right)$$

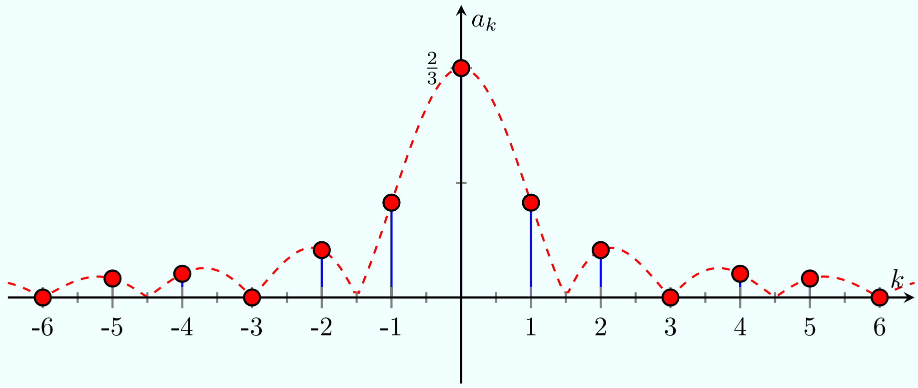
$$a_k = \frac{1}{T_0}\frac{2\sin(k\omega_0 T_1/2)}{k\omega_0}$$

Since $\omega_0 = \frac{2\pi}{T_0}$,

$$a_k = \frac{\sin(\pi k T_1/T_0)}{\pi k}$$

Given $\frac{T_1}{T_0} = \frac{2}{3}$,

$$a_k = \frac{\sin(2\pi k/3)}{\pi k}$$



(e) Evaluating the Fourier transform at $\omega = \frac{2\pi k}{T_0}$,

$$X\left(\frac{2\pi k}{T_0}\right) = \frac{T_0}{\pi k}\sin\left(\pi k\frac{T_1}{T_0}\right)$$

(f) Hence the Fourier series coefficients are scaled samples of the Fourier transform:

$$a_k = \frac{1}{T_0} X\left(\frac{2\pi k}{T_0}\right)$$

$$\boxed{X(\omega) = \frac{2\sin(\omega T_1/2)}{\omega}, \quad a_k = \frac{\sin(2\pi k/3)}{\pi k}, \quad a_k = \frac{1}{T_0} X\left(\frac{2\pi k}{T_0}\right)}$$

Problem 2

Determine the Fourier transform of $x(t) = e^{-t/2}u(t)$ and sketch:

(a) $|X(\omega)|$

(b) $\angle X(\omega)$

(c) $\text{Re}\{X(\omega)\}$

(d) $\text{Im}\{X(\omega)\}$

Solution: Given $x(t) = e^{-t/2}u(t)$:

$$X(\omega) = \int_0^\infty e^{-t/2} e^{-j\omega t} dt = \int_0^\infty e^{-(1/2+j\omega)t} dt = \frac{1}{1/2 + j\omega} = \frac{2}{1 + j2\omega}$$

Writing in terms of real and imaginary parts:

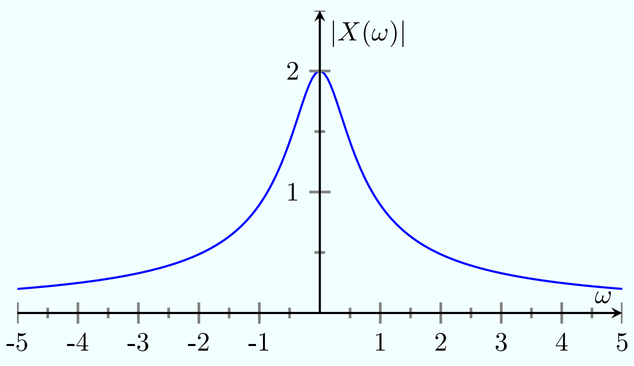
$$X(\omega) = \frac{2}{1+j2\omega} \cdot \frac{1-j2\omega}{1-j2\omega} = \frac{2(1-j2\omega)}{1+4\omega^2} = \frac{2}{1+4\omega^2} - j\frac{4\omega}{1+4\omega^2}$$

Thus:

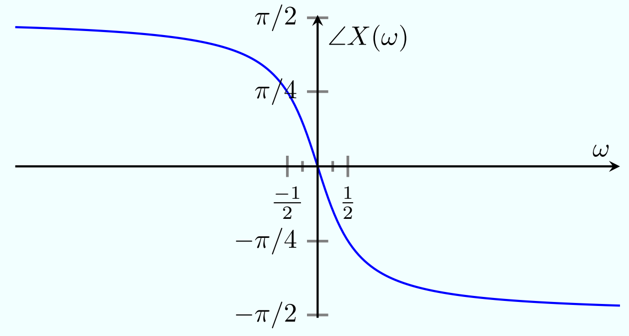
$$|X(\omega)| = \frac{2}{\sqrt{1+4\omega^2}}, \quad \angle X(\omega) = -\tan^{-1}(2\omega)$$

$$\text{Re}\{X(\omega)\} = \frac{2}{1+4\omega^2}, \quad \text{Im}\{X(\omega)\} = -\frac{4\omega}{1+4\omega^2}$$

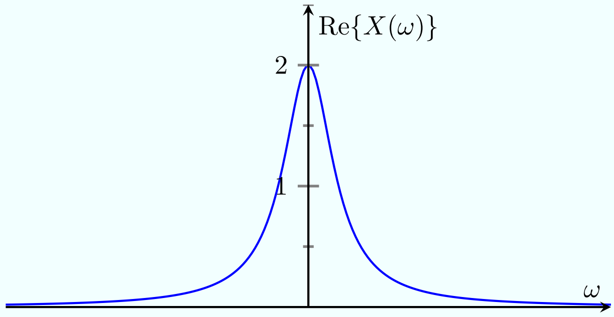
(a) Magnitude



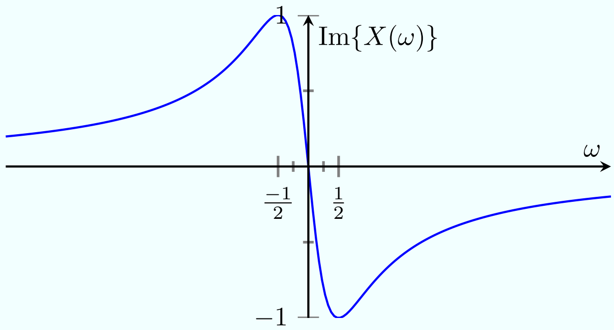
(b) Phase



(c) Real part



(d) Imaginary part



$$\boxed{X(\omega) = \frac{2}{1 + j2\omega}}$$

Problem 3

(a) By considering the Fourier analysis equation or synthesis equation, show the validity in general of each of the following statements:

(i) If $x(t)$ is real-valued, then $X(\omega) = X^*(-\omega)$.

(ii) If $x(t) = x^*(-t)$, then $X(\omega)$ is real-valued.

(b) Using the statements in part (a), show the validity of each of the following statements:

(i) If $x(t)$ is real and even, then $X(\omega)$ is real and even.

(ii) If $x(t)$ is real and odd, then $X(\omega)$ is imaginary and odd.

Solution:

(a) (i) $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$

Taking complex conjugate:

$$X^*(\omega) = \int_{-\infty}^{\infty} x^*(t)e^{j\omega t}dt$$

Since $x(t)$ is real, $x^*(t) = x(t)$:

$$X^*(\omega) = \int_{-\infty}^{\infty} x(t)e^{j\omega t}dt$$

Therefore:

$$X^*(-\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = X(\omega)$$

(ii) $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$

Taking complex conjugate:

$$x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)e^{-j\omega t}d\omega$$

Replacing t with $-t$:

$$x^*(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)e^{j\omega t}d\omega$$

Given $x(t) = x^*(-t)$:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)e^{j\omega t}d\omega$$

This implies $X(\omega) = X^*(\omega)$, so $X(\omega)$ is real.

(b) (i) $x(t)$ real: $X(\omega) = X^*(-\omega)$ $x(t)$ even: $x(t) = x(-t)$, and since real, $x(t) = x^*(-t)$ From (a)(ii), $X(\omega)$ is real. Then $X(\omega) = X^*(-\omega) = X(-\omega)$, so $X(\omega)$ is even.

(ii) $x(t)$ real: $X(\omega) = X^*(-\omega)$ $x(t)$ odd: $x(t) = -x(-t)$, and since real, $x(t) = -x^*(-t)$ Following similar analysis as (a)(ii) but with negative sign, $X(\omega)$ must be imaginary. Then $X(\omega) = X^*(-\omega) = -X(-\omega)$, so $X(\omega)$ is odd.

$$\boxed{\text{Properties proven as above}}$$

Problem 4

The output of a causal LTI system is related to the input $x(t)$ by the differential equation:

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

(a) Determine the frequency response $H(\omega) = Y(\omega)/X(\omega)$ and sketch the phase and magnitude of $H(\omega)$.

(b) If $x(t) = e^{-t}u(t)$, determine $Y(\omega)$, the Fourier transform of the output.

(c) Find $y(t)$ for the input given in part (b).

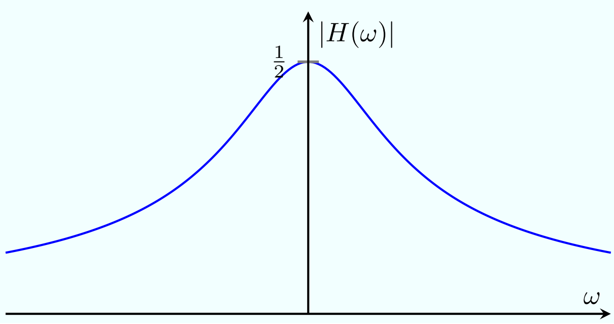
Solution:

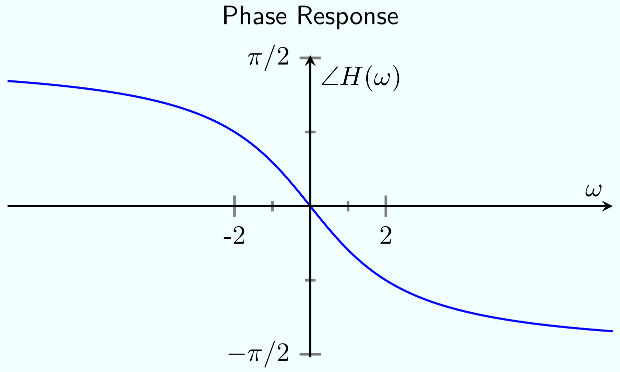
(a) Taking Fourier transform of the differential equation:

$$j\omega Y(\omega) + 2Y(\omega) = X(\omega)$$
$$Y(\omega)[2 + j\omega] = X(\omega)$$
$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{2 + j\omega}$$

$$|H(\omega)| = \frac{1}{\sqrt{4+\omega^2}}, \quad \angle H(\omega) = -\tan^{-1}\left(\frac{\omega}{2}\right)$$

Magnitude Response





(b) $x(t) = e^{-t}u(t)$, so $X(\omega) = \frac{1}{1+j\omega}$

$$Y(\omega) = H(\omega)X(\omega) = \frac{1}{(2+j\omega)(1+j\omega)}$$

(c) Using partial fractions:

$$\frac{1}{(2+j\omega)(1+j\omega)} = \frac{1}{1+j\omega} - \frac{1}{2+j\omega}$$

Taking inverse Fourier transform:

$$y(t) = e^{-t}u(t) - e^{-2t}u(t)$$

$$\boxed{H(\omega) = \frac{1}{2+j\omega}, \quad Y(\omega) = \frac{1}{(2+j\omega)(1+j\omega)}, \quad y(t) = (e^{-t} - e^{-2t})u(t)}$$

Problem 5

Compute the Fourier transform of each of the following signals:

(a) $[e^{-at}\cos\omega_0 t]u(t)$, $a > 0$

(b) $e^{-3|t|}\sin 2t$

(c) $\frac{\sin \pi t}{\pi t} \cdot \frac{\sin 2\pi t}{\pi t}$

Solution:

(a) $x(t) = e^{-at}\cos(\omega_0 t)u(t) = e^{-at}u(t) \cdot \cos(\omega_0 t)$

$\mathcal{F}\{e^{-at}u(t)\} = \frac{1}{a+j\omega}$

$\mathcal{F}\{\cos(\omega_0 t)\} = \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$

Using multiplication in time = convolution in frequency (scaled by $1/(2\pi)$):

$$X(\omega) = \frac{1}{2\pi}\left[\frac{1}{a+j\omega} * \pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))\right]$$

$$X(\omega) = \frac{1}{2}\left[\frac{1}{a+j(\omega-\omega_0)} + \frac{1}{a+j(\omega+\omega_0)}\right]$$

(b) $x(t) = e^{-3|t|}\sin 2t$

$\mathcal{F}\{e^{-3|t|}\} = \frac{6}{9+\omega^2}$

$\mathcal{F}\{\sin 2t\} = \frac{\pi}{j}[\delta(\omega - 2) - \delta(\omega + 2)]$

Using multiplication property:

$$X(\omega) = \frac{1}{2\pi}\left[\frac{6}{9+\omega^2} * \frac{\pi}{j}(\delta(\omega-2) - \delta(\omega+2))\right]$$

$$X(\omega) = \frac{3}{j}\left[\frac{1}{9+(\omega-2)^2} - \frac{1}{9+(\omega+2)^2}\right]$$

(c) $x(t) = \frac{\sin \pi t}{\pi t} \cdot \frac{\sin 2\pi t}{\pi t}$

$\mathcal{F}\left\{\frac{\sin \pi t}{\pi t}\right\} = \text{rect}\left(\frac{\omega}{2\pi}\right) = \begin{cases} 1, & |\omega| < \pi \\ 0, & |\omega| > \pi \end{cases}$

$\mathcal{F}\left\{\frac{\sin 2\pi t}{\pi t}\right\} = \text{rect}\left(\frac{\omega}{4\pi}\right) = \begin{cases} 1, & |\omega| < 2\pi \\ 0, & |\omega| > 2\pi \end{cases}$

Using multiplication property:

$$X(\omega) = \frac{1}{2\pi}[\text{rect}(\omega/(2\pi)) * \text{rect}(\omega/(4\pi))]$$

The convolution of two rectangles results in a trapezoid:

$$X(\omega) = \begin{cases} \frac{1}{2\pi}(\omega + 3\pi), & -3\pi < \omega < -\pi \\ \frac{1}{2\pi}(2\pi), & -\pi < \omega < \pi \\ \frac{1}{2\pi}(3\pi - \omega), & \pi < \omega < 3\pi \\ 0, & \text{otherwise} \end{cases}$$

$$\boxed{X_a(\omega) = \frac{1}{2}\left[\frac{1}{a + j(\omega - \omega_0)} + \frac{1}{a + j(\omega + \omega_0)}\right]}$$

$$\boxed{X_b(\omega) = \frac{3}{j}\left[\frac{1}{9 + (\omega - 2)^2} - \frac{1}{9 + (\omega + 2)^2}\right]}$$

$$\boxed{X_c(\omega) = \frac{1}{2\pi}\text{rect}(\omega/(2\pi)) * \text{rect}(\omega/(4\pi))}$$

Problem 6

Use properties of the Fourier transform to show by induction that the Fourier transform of

$$x(t) = \frac{t^{n-1}}{(n-1)!}e^{-at}u(t), \quad a > 0$$

is

$$X(\omega) = \frac{1}{(a + j\omega)^n}$$

Solution: We prove by induction.

Base case: $n = 1$

$$x(t) = e^{-at}u(t), \quad X(\omega) = \frac{1}{a + j\omega} = \frac{1}{(a + j\omega)^1}$$

Inductive step: Assume true for n:

$$x_n(t) = \frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \quad \Leftrightarrow \quad X_n(\omega) = \frac{1}{(a + j\omega)^n}$$

Consider $n + 1$:

$$x_{n+1}(t) = \frac{t^n}{n!}e^{-at}u(t) = \frac{t}{n} \cdot \frac{t^{n-1}}{(n-1)!}e^{-at}u(t) = \frac{t}{n}x_n(t)$$

Using the differentiation in frequency property: $\mathcal{F}\{tx_n(t)\} = j\frac{d}{d\omega}X_n(\omega)$

Therefore:

$$\mathcal{F}\{x_{n+1}(t)\} = \frac{1}{n} \cdot j\frac{d}{d\omega}X_n(\omega) = \frac{j}{n} \cdot \frac{d}{d\omega}\left[\frac{1}{(a + j\omega)^n}\right]$$

Computing the derivative:

$$\frac{d}{d\omega}\left[\frac{1}{(a + j\omega)^n}\right] = \frac{-n \cdot j}{(a + j\omega)^{n+1}}$$

Thus:

$$\mathcal{F}\{x_{n+1}(t)\} = \frac{j}{n} \cdot \frac{-n \cdot j}{(a + j\omega)^{n+1}} = \frac{1}{(a + j\omega)^{n+1}}$$

This completes the induction.

$$\boxed{\mathcal{F}\left\{\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)\right\} = \frac{1}{(a+j\omega)^n}}$$

Problem 7

In this problem, we explore the definition of the Fourier transform of a periodic signal.

(a) Show that if $x_3(t) = ax_1(t) + bx_2(t)$, then $X_3(\omega) = aX_1(\omega) + bX_2(\omega)$.

(b) Verify that

$$e^{j\omega_0 t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}d\omega$$

From this observation, argue that the Fourier transform of $e^{j\omega_0 t}$ is $2\pi\delta(\omega - \omega_0)$.

(c) Recall the synthesis equation for the Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

By taking the Fourier transform of both sides and using the results to parts (a) and (b), show that

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

Solution:

(a)

$$X_3(\omega) = \int_{-\infty}^{\infty} [ax_1(t) + bx_2(t)]e^{-j\omega t}dt$$
$$= a\int_{-\infty}^{\infty} x_1(t)e^{-j\omega t}dt + b\int_{-\infty}^{\infty} x_2(t)e^{-j\omega t}dt$$
$$= aX_1(\omega) + bX_2(\omega)$$

(b) By the sifting property of the impulse:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}d\omega = \int_{-\infty}^{\infty} \delta(\omega - \omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t}$$

This is exactly the inverse Fourier transform formula:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

with $X(\omega) = 2\pi\delta(\omega - \omega_0)$. Therefore, $\mathcal{F}\{e^{j\omega_0 t}\} = 2\pi\delta(\omega - \omega_0)$.

(c) Taking Fourier transform of both sides of the Fourier series synthesis equation:

$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right\}$$

Using linearity from part (a):

$$X(\omega) = \sum_{k=-\infty}^{\infty} a_k \mathcal{F}\{e^{jk\omega_0 t}\}$$

Using result from part (b) with ω_0 replaced by $k\omega_0$:

$$X(\omega) = \sum_{k=-\infty}^{\infty} a_k \cdot 2\pi\delta(\omega - k\omega_0)$$

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

$$\boxed{X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)}$$

— End of Problem Set —