LUMS
A Not-for-Profit University

School of Science and Engineering

# EE 310 Signals and Systems

# PROBLEM SET 6

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## Problem 1

Let $m(t)$ be a real message signal whose Fourier transform $M(j\omega)$ is bandlimited to $|\omega| \le W$. The DSB-SC modulated signal is

$$s(t) = m(t)\cos(\omega_c t + \theta), \qquad \omega_c \gg W,$$

where $\theta$ is an unknown constant phase offset.

1. Derive $S(j\omega)$ in terms of $M(j\omega)$, $\omega_c$, and $\theta$.
2. A coherent demodulator multiplies $s(t)$ by the phase-mismatched local carrier $2\cos(\omega_c t)$ and passes the result through an ideal low-pass filter with cutoff $\omega_c$ and unit gain. Show that the recovered signal is $m(t)\cos\theta$, and identify all values of $\theta \in [0, 2\pi)$ for which the demodulator *completely suppresses* the output.
3. Now suppose the phase offset is time-varying: $\theta(t) = \alpha t$ for some small $\alpha > 0$. Characterize the spectrum of the demodulated output and determine the maximum $\alpha$ for which $m(t)$ can still be recovered without spectral overlap distortion.
4. Suppose $m(t) = \sum_{k=1}^{N} A_k \cos(\omega_k t)$ with $\omega_k \le W$ for all $k$. Write an explicit closed-form expression for $S(j\omega)$ as a sum of impulses when $\theta = \pi/4$.

## Problem 2

Figure (c) shows a remarkable phenomenon: a smooth continuous signal $g(t)$ (solid curve) can be *exactly* reconstructed by summing infinitely many scaled and shifted copies of a single function. Each vertical arrow represents a measurement of $g(t)$ taken at equally spaced instants $t = nT_s$; the dashed curves are the resulting scaled copies. This problem guides you through the mathematics behind this picture using only the CTFT tools you have already learned.

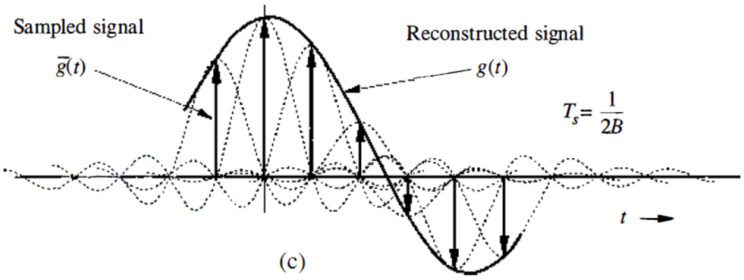

Figure (c): A signal $g(t)$ reconstructed as a superposition of scaled, shifted sinc functions centred at the sample instants $t = nT_s$.

1. **The reconstruction building block.** Consider the ideal low-pass filter with frequency

response

$$H(j\omega) = \begin{cases} T_s, & |\omega| \le \dfrac{\pi}{T_s}, \\ 0, & |\omega| > \dfrac{\pi}{T_s}. \end{cases}$$

(i) Compute the impulse response $h(t)$ by taking the inverse CTFT of $H(j\omega)$. Show that

$$h(t) = \operatorname{sinc}\left(\frac{t}{T_s}\right) \equiv \frac{\sin(\pi t/T_s)}{\pi t/T_s}.$$

(ii) Evaluate $h(nT_s)$ for every integer $n$. What do you notice? Explain in one sentence why this property is essential for reconstruction.

2. **A single sample.** Suppose only one measurement is taken at $t = 0$, giving the value $g(0)$. We represent this measurement as the signal

$$\bar{g}_0(t) = g(0)\,\delta(t).$$

(i) Find the CTFT of $\bar{g}_0(t)$.

(ii) Pass $\bar{g}_0(t)$ through the filter $H(j\omega)$ above. Find the output $y_0(t)$ in the time domain and sketch it.

(iii) How does the peak value and location of $y_0(t)$ relate to the original sample $g(0)$?

3. **Two samples.** Now two measurements are taken, at $t = 0$ and $t = T_s$, giving the signal

$$\bar{g}_{01}(t) = g(0)\,\delta(t) + g(T_s)\,\delta(t - T_s).$$

(i) Use the linearity and time-shift properties of the CTFT to find $\bar{G}_{01}(j\omega)$.

(ii) Pass $\bar{g}_{01}(t)$ through $H(j\omega)$ and show that the output is

$$y_{01}(t) = g(0)\operatorname{sinc}\left(\frac{t}{T_s}\right) + g(T_s)\operatorname{sinc}\left(\frac{t - T_s}{T_s}\right).$$

(iii) Evaluate $y_{01}(t)$ at $t = 0$ and $t = T_s$. Verify that the output *exactly* passes through both sample values, and explain using part (1)(ii) why the two sincs do not interfere with each other at the sample instants.

4. **Infinitely many samples (the full picture).** Generalising parts (2) and (3), suppose measurements are taken at every instant $t = nT_s$ for $n \in \mathbb{Z}$, giving

$$\bar{g}(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\,\delta(t - nT_s).$$

(i) Use linearity and the time-shift property to write down $\bar{G}(j\omega)$ directly. (No new calculation is needed — quote the pattern from part (3)(i).)

(ii) Pass $\bar{g}(t)$ through $H(j\omega)$ and write the output $g_r(t)$ as an explicit infinite sum of sinc functions. This is the formula corresponding to Figure (c).

(iii) From your expression for $g_r(t)$, read off $g_r(nT_s)$ for any integer $n$ and confirm that the reconstruction *interpolates* the original samples exactly.

(iv) Identify which curves in Figure (c) correspond to (a) the individual sinc terms in your sum, (b) the sample values $g(nT_s)$, and (c) the reconstructed signal $g_r(t)$.

## Problem 3

A causal LTI system is described by the LCDE

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6\,y(t) = \frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3\,x(t).$$

1. Find the frequency response $H(j\omega)$ and factor both numerator and denominator completely over $\mathbb{C}$.
2. Perform a partial fraction decomposition of $H(j\omega)$ and hence compute the impulse response $h(t)$.

3. Determine $|H(j\omega)|$ and $\angle H(j\omega)$. Find the 3-dB bandwidth.
4. The input $x(t) = e^{-t}u(t) + \delta(t)$ is applied. Compute $Y(j\omega)$ and find $y(t)$ in closed form.
5. Determine BIBO stability from the poles of $H(j\omega)$, then verify by showing directly from $h(t)$ that $\displaystyle\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$.

## Problem 4

Recall the CTFT duality principle: if $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$, then $X(jt) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\, x(-\omega)$.

1. Given that $\dfrac{1}{1+t^2} \overset{\mathcal{F}}{\longleftrightarrow} \pi e^{-|\omega|}$, use duality to derive the Fourier transform of $e^{-|t|}$ *without direct integration.*
2. Given $\Pi(t/\tau) \overset{\mathcal{F}}{\longleftrightarrow} \tau\, \mathrm{sinc}(\omega\tau/2)$, apply duality to find the transform of $W\,\mathrm{sinc}(Wt/2)$ and interpret the result in terms of ideal low-pass filtering.
3. Let $X(j\omega) = \Lambda(\omega/2W)$ be a triangular spectrum. Using duality and the convolution theorem, express $x(t)$ as the square of a sinc function.
4. A signal $g(t)$ satisfies $g(t) = G(jt)$ (self-dual). Using only Fourier transform properties, show that $|G(j\omega)|^2 = 2\pi\, |g(\omega)|^2$ and explain what constraint this places on the total energy of $g$.

## Problem 5

Two bandlimited signals $m_1(t)$ and $m_2(t)$, each with bandwidth $W$, are transmitted simultaneously using *quadrature amplitude modulation*:

$$s(t) = m_1(t)\cos(\omega_c t) - m_2(t)\sin(\omega_c t), \qquad \omega_c > W.$$

1. Derive $S(j\omega)$ in terms of $M_1(j\omega)$ and $M_2(j\omega)$.
2. Describe a complete coherent demodulation scheme that recovers $m_1(t)$ and $m_2(t)$ *separately* from $s(t)$. Prove using the CTFT that there is zero cross-channel interference.
3. The channel introduces a phase error: the received signal is $r(t) = s(t)$ but the local carrier has a phase offset $\phi$, so the demodulator uses $2\cos(\omega_c t + \phi)$ and $-2\sin(\omega_c t + \phi)$. Find the recovered signals in terms of $m_1$, $m_2$, and $\phi$. For what value of $\phi$ are the two channels completely swapped?
4. Verify using Parseval's theorem and the result of part (a) that the total transmitted average power satisfies $P_s = \frac{1}{2}(P_{m_1} + P_{m_2})$.

## Problem 6

Two causal LTI subsystems are described by:

$$H_1 : \quad \frac{dy_1}{dt} + 3\,y_1(t) = x(t), \qquad H_2 : \quad \frac{d^2y_2}{dt^2} + 2\frac{dy_2}{dt} + 5\,y_2(t) = \frac{dy_1}{dt} + y_1(t),$$

where the output of $H_1$ drives $H_2$.

1. Find $H_1(j\omega)$, $H_2(j\omega)$, and the overall frequency response $H(j\omega) = H_1(j\omega)\,H_2(j\omega)$.
2. Compute the impulse response $h(t)$ of the overall system using partial fractions.
3. Classify $H(j\omega)$ as low-pass, high-pass, or band-pass by evaluating $|H(j\omega)|$ at $\omega = 0$, $\omega \to \infty$, and the peak frequency $\omega^*$.
4. The system is now changed to the following: $H_{\text{fb}}(j\omega) = H(j\omega)/[1 + H(j\omega)]$. Find the poles of $H_{\text{fb}}(j\omega)$ and determine whether the system is BIBO stable. (This is a negative feedback loop that has a unity gain. You will learn more about this in your Feedback Control Systems Course :) )
5. For the input $x(t) = \cos(2t)\,u(t)$, find the *steady-state* component of $y_2(t)$ as $t \to \infty$ using the frequency response directly.

## Problem 7

Define the signal

$$p(t) = \frac{\sin(Wt)}{\pi t} \cdot \frac{\sin(Bt)}{\pi t}, \qquad 0 < B < W.$$

1. Using the convolution theorem and the duality principle, find $P(j\omega)$ — the CTFT of $p(t)$ — *without computing any integral.* Sketch $P(j\omega)$ and identify it as a named standard spectral shape.
2. $p(t)$ is used as the impulse response of a filter applied to the modulated signal $s(t) = m(t)\cos(\omega_c t)$, where $m(t)$ is bandlimited to $W$ and $\omega_c \gg W$. Assuming $\omega_c - W > B$, find the CTFT of the output $y(t) = (p * s)(t)$ and describe the filtering action in plain terms.
3. Consider the LCC differential equation

   $$\frac{d^2y}{dt^2} + W^2 y(t) = p(t).$$

   Take the CTFT of both sides and find $Y(j\omega)$. Identify any frequency components of $P(j\omega)$ that coincide with the natural frequency of the system and explain what phenomenon arises.
4. Apply the duality principle to $P(j\omega)$ found in part (a) to determine the CTFT of the time-domain signal $P(jt)$. Express the result as a closed-form signal and verify consistency with part (a).

— End of Problem Set —