LUMS
A Not-for-Profit University

School of Science and Engineering

EE 310 Signals and Systems

PROBLEM SET 6 – SOLUTIONS

Problem 1

Let $m(t)$ be a real message signal whose Fourier transform $M(j\omega)$ is bandlimited to $|\omega| \le W$. The DSB-SC modulated signal is

$$s(t) = m(t)\cos(\omega_c t + \theta), \qquad \omega_c \gg W,$$

where θ is an unknown constant phase offset.

1. Derive $S(j\omega)$ in terms of $M(j\omega)$, ω_c, and θ.
2. A coherent demodulator multiplies $s(t)$ by the phase-mismatched local carrier $2\cos(\omega_c t)$ and passes the result through an ideal low-pass filter with cutoff ω_c and unit gain. Show that the recovered signal is $m(t)\cos\theta$, and identify all values of $\theta \in [0, 2\pi)$ for which the demodulator *completely suppresses* the output.
3. Now suppose the phase offset is time-varying: $\theta(t) = \alpha t$ for some small $\alpha > 0$. Characterize the spectrum of the demodulated output and determine the maximum α for which $m(t)$ can still be recovered without spectral overlap distortion.
4. Suppose $m(t) = \sum_{k=1}^{N} A_k \cos(\omega_k t)$ with $\omega_k \le W$ for all k. Write an explicit closed-form expression for $S(j\omega)$ as a sum of impulses when $\theta = \pi/4$.

Solution:

1. Using Euler's formula $\cos(\omega_c t + \theta) = \frac{1}{2}(e^{j(\omega_c t+\theta)} + e^{-j(\omega_c t+\theta)})$:
$$S(j\omega) = \frac{1}{2}\big[e^{j\theta}M(j(\omega-\omega_c)) + e^{-j\theta}M(j(\omega+\omega_c))\big].$$
2. After multiplication by $2\cos(\omega_c t)$:
$$v(t) = 2s(t)\cos(\omega_c t) = m(t)\big[\cos\theta + \cos(2\omega_c t + \theta)\big].$$
The LPF removes the $2\omega_c$ term, yielding $y(t) = m(t)\cos\theta$. The output is suppressed when $\cos\theta = 0$, i.e. $\theta = \pi/2$ or $\theta = 3\pi/2$.
3. With $\theta(t) = \alpha t$, $s(t) = m(t)\cos((\omega_c+\alpha)t)$. The demodulated spectrum is $\frac{1}{2}[M(j(\omega-\alpha)) + M(j(\omega+\alpha))]$. For no spectral overlap, the shifted copies must not exceed the LPF cutoff:
$$\alpha < \omega_c - W.$$
4. With $\theta = \pi/4$ and $m(t) = \sum_k A_k\cos(\omega_k t)$:
$$S(j\omega) = \frac{\pi}{\sqrt{2}}\sum_{k=1}^{N} A_k\Big[\delta(\omega-\omega_c-\omega_k) + \delta(\omega-\omega_c+\omega_k) + \delta(\omega+\omega_c-\omega_k) + \delta(\omega+\omega_c+\omega_k)\Big].$$

$$\boxed{S(j\omega) = \tfrac{1}{2}\big[e^{j\theta}M(j(\omega-\omega_c)) + e^{-j\theta}M(j(\omega+\omega_c))\big], \quad y(t) = m(t)\cos\theta, \quad \alpha_{\max} = \omega_c - W}$$

Problem 2

Figure (c) shows a remarkable phenomenon: a smooth continuous signal $g(t)$ (solid curve) can be *exactly* reconstructed by summing infinitely many scaled and shifted copies of a single function. Each vertical arrow represents a measurement of $g(t)$ taken at equally spaced instants $t = nT_s$; the dashed curves are the resulting scaled copies. This problem guides you through the mathematics behind this picture using only the CTFT tools you have already learned.

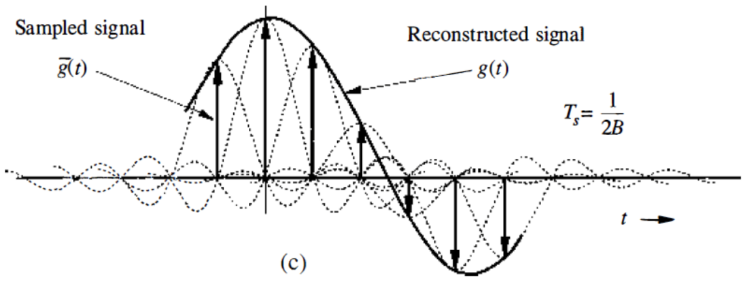

Figure (c): A signal $g(t)$ reconstructed as a superposition of scaled, shifted sinc functions centred at the sample instants $t = nT_s$.

1. **The reconstruction building block.** Consider the ideal low-pass filter with frequency response

$$H(j\omega) = \begin{cases} T_s, & |\omega| \le \dfrac{\pi}{T_s}, \\ 0, & |\omega| > \dfrac{\pi}{T_s}. \end{cases}$$

   (i) Compute the impulse response $h(t)$ by taking the inverse CTFT of $H(j\omega)$. Show that

$$h(t) = \operatorname{sinc}\left(\frac{t}{T_s}\right) \equiv \frac{\sin(\pi t/T_s)}{\pi t/T_s}.$$

   (ii) Evaluate $h(nT_s)$ for every integer n. What do you notice? Explain in one sentence why this property is essential for reconstruction.

2. **A single sample.** Suppose only one measurement is taken at $t = 0$, giving the value $g(0)$. We represent this measurement as the signal

$$\bar{g}_0(t) = g(0)\,\delta(t).$$

   (i) Find the CTFT of $\bar{g}_0(t)$.

   (ii) Pass $\bar{g}_0(t)$ through the filter $H(j\omega)$ above. Find the output $y_0(t)$ in the time domain and sketch it.

   (iii) How does the peak value and location of $y_0(t)$ relate to the original sample $g(0)$?

3. **Two samples.** Now two measurements are taken, at $t = 0$ and $t = T_s$, giving the signal

$$\bar{g}_{01}(t) = g(0)\,\delta(t) + g(T_s)\,\delta(t - T_s).$$

   (i) Use the linearity and time-shift properties of the CTFT to find $\bar{G}_{01}(j\omega)$.

   (ii) Pass $\bar{g}_{01}(t)$ through $H(j\omega)$ and show that the output is

$$y_{01}(t) = g(0)\operatorname{sinc}\left(\frac{t}{T_s}\right) + g(T_s)\operatorname{sinc}\left(\frac{t - T_s}{T_s}\right).$$

   (iii) Evaluate $y_{01}(t)$ at $t = 0$ and $t = T_s$. Verify that the output *exactly* passes through both sample values, and explain using part (1)(ii) why the two sincs do not interfere with each other at the sample instants.

4. **Infinitely many samples (the full picture).** Generalising parts (2) and (3), suppose measurements are taken at every instant $t = nT_s$ for $n \in \mathbb{Z}$, giving

$$\bar{g}(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\,\delta(t - nT_s).$$

(i) Use linearity and the time-shift property to write down $\bar{G}(j\omega)$ directly. (No new calculation is needed — quote the pattern from part (3)(i).)

(ii) Pass $\bar{g}(t)$ through $H(j\omega)$ and write the output $g_r(t)$ as an explicit infinite sum of sinc functions. This is the formula corresponding to Figure (c).

(iii) From your expression for $g_r(t)$, read off $g_r(nT_s)$ for any integer n and confirm that the reconstruction *interpolates* the original samples exactly.

(iv) Identify which curves in Figure (c) correspond to (a) the individual sinc terms in your sum, (b) the sample values $g(nT_s)$, and (c) the reconstructed signal $g_r(t)$.

Solution:

1. **The reconstruction building block.**

   (i) Inverse CTFT:
   $$h(t) = \frac{1}{2\pi}\int_{-\pi/T_s}^{\pi/T_s} T_s\, e^{j\omega t}\, d\omega = \frac{T_s}{2\pi}\cdot\frac{2\sin(\pi t/T_s)}{t} = \frac{\sin(\pi t/T_s)}{\pi t/T_s} = \text{sinc}\left(\frac{t}{T_s}\right).\checkmark$$

   (ii) $h(nT_s) = \text{sinc}(n) = \delta[n]$: it equals 1 when $n = 0$ and 0 for all other integers. This means a sinc centred at one sample instant contributes *nothing* to any other sample instant, so the sincs do not corrupt each other's values.

2. **A single sample.**

   (i) $\bar{G}_0(j\omega) = g(0)$ (a constant, flat spectrum).

   (ii) $Y_0(j\omega) = H(j\omega)\cdot g(0)$, so $y_0(t) = g(0)\, h(t) = g(0)\,\text{sinc}(t/T_s)$: a sinc centred at $t = 0$ with peak value $g(0)$.

   (iii) The peak of $y_0(t)$ occurs at $t = 0$ with value $g(0)$, exactly matching the original sample.

3. **Two samples.**

   (i) By linearity and the time-shift property:
   $$\bar{G}_{01}(j\omega) = g(0) + g(T_s)\, e^{-j\omega T_s}.$$

   (ii) $Y_{01}(j\omega) = H(j\omega)\bar{G}_{01}(j\omega)$; by linearity and the shift property applied to $h(t)$:
   $$y_{01}(t) = g(0)\,\text{sinc}\left(\frac{t}{T_s}\right) + g(T_s)\,\text{sinc}\left(\frac{t-T_s}{T_s}\right).\checkmark$$

   (iii) At $t = 0$: $\text{sinc}(0) = 1$ and $\text{sinc}(-1) = 0$, so $y_{01}(0) = g(0)$. At $t = T_s$: $\text{sinc}(1) = 0$ and $\text{sinc}(0) = 1$, so $y_{01}(T_s) = g(T_s)$. Both sample values are recovered exactly because $\text{sinc}(n) = \delta[n]$ (part 1(ii)).

4. **Infinitely many samples.**

   (i) By linearity and time-shift:
   $$\bar{G}(j\omega) = \sum_{n=-\infty}^{\infty} g(nT_s)\, e^{-j\omega nT_s}.$$

   (ii) Multiplying by $H(j\omega)$ in frequency $\equiv$ convolving with $h(t)$:
   $$g_r(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\ \text{sinc}\left(\frac{t-nT_s}{T_s}\right).$$

   (iii) $g_r(mT_s) = \sum_n g(nT_s)\,\text{sinc}(m-n) = \sum_n g(nT_s)\delta[m-n] = g(mT_s)$. ✓

   (iv) In Figure (c): (a) the dashed curves are the individual sinc terms; (b) the vertical arrows are the sample values $g(nT_s)$; (c) the bold solid curve is the reconstructed signal $g_r(t) = g(t)$.

$$\boxed{h(t) = \text{sinc}\left(\tfrac{t}{T_s}\right),\quad h(nT_s) = \delta[n],\quad g_r(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\,\text{sinc}\left(\tfrac{t-nT_s}{T_s}\right)}$$

Problem 3

A causal LTI system is described by the LCDE

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6\,y(t) = \frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3\,x(t).$$

1. Find the frequency response $H(j\omega)$ and factor both numerator and denominator completely over $\mathbb{C}$.
2. Perform a partial fraction decomposition of $H(j\omega)$ and hence compute the impulse response $h(t)$.
3. Determine $|H(j\omega)|$ and $\angle H(j\omega)$. Find the 3-dB bandwidth.
4. The input $x(t) = e^{-t}u(t) + \delta(t)$ is applied. Compute $Y(j\omega)$ and find $y(t)$ in closed form.
5. Determine BIBO stability from the poles of $H(j\omega)$, then verify by showing directly from $h(t)$ that $\displaystyle\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$.

Solution:

1. Taking the CTFT:
$$H(j\omega) = \frac{(j\omega)^2 + 4(j\omega) + 3}{(j\omega)^3 + 6(j\omega)^2 + 11(j\omega) + 6} = \frac{(j\omega+1)(j\omega+3)}{(j\omega+1)(j\omega+2)(j\omega+3)}.$$
After cancellation: $H(j\omega) = \dfrac{1}{j\omega+2}$.
2. Since $H(j\omega) = \dfrac{1}{j\omega+2}$, no further PFD is needed. $h(t) = e^{-2t}u(t)$.
3. $|H(j\omega)| = \dfrac{1}{\sqrt{\omega^2+4}}$, $\angle H(j\omega) = -\arctan(\omega/2)$. 3-dB bandwidth: $|H|^2 = \frac{1}{2}|H(0)|^2 \Rightarrow \omega_{3\text{dB}} = 2$ rad/s.
4. $X(j\omega) = \frac{1}{j\omega+1} + 1$. Thus $Y(j\omega) = \frac{1}{(j\omega+1)(j\omega+2)} + \frac{1}{j\omega+2}$. PFD of first term: $\frac{1}{j\omega+1} - \frac{1}{j\omega+2}$. So:
$$y(t) = \left(e^{-t} - e^{-2t}\right)u(t) + e^{-2t}u(t) = e^{-t}u(t).$$
5. The single pole at $j\omega = -2$ (i.e. $s = -2$) lies in the left half-plane $\Rightarrow$ BIBO stable. $\int_0^\infty e^{-2t}\,dt = \frac{1}{2} < \infty$. ✓

$$\boxed{H(j\omega) = \frac{1}{j\omega+2}, \quad h(t) = e^{-2t}u(t), \quad \omega_{3\text{dB}} = 2 \text{ rad/s}, \quad y(t) = e^{-t}u(t), \quad \text{BIBO stable}}$$

Problem 4

Recall the CTFT duality principle: if $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$, then $X(jt) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\,x(-\omega)$.

1. Given that $\dfrac{1}{1+t^2} \overset{\mathcal{F}}{\longleftrightarrow} \pi e^{-|\omega|}$, use duality to derive the Fourier transform of $e^{-|t|}$ *without direct integration.*
2. Given $\Pi(t/\tau) \overset{\mathcal{F}}{\longleftrightarrow} \tau\,\text{sinc}(\omega\tau/2)$, apply duality to find the transform of $W\,\text{sinc}(Wt/2)$ and interpret the result in terms of ideal low-pass filtering.
3. Let $X(j\omega) = \Lambda(\omega/2W)$ be a triangular spectrum. Using duality and the convolution theorem, express $x(t)$ as the square of a sinc function.
4. A signal $g(t)$ satisfies $g(t) = G(jt)$ (self-dual). Using only Fourier transform properties, show that $|G(j\omega)|^2 = 2\pi\,|g(\omega)|^2$ and explain what constraint this places on the total energy of g.

Solution:

1. Let $x(t) = \frac{1}{1+t^2}$, $X(j\omega) = \pi e^{-|\omega|}$. Duality: $X(jt) = \pi e^{-|t|} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi x(-\omega) = \frac{2\pi}{1+\omega^2}$. Dividing both sides by π:
$$e^{-|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2}{1+\omega^2}.$$
2. $W\,\mathrm{sinc}(Wt/2) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\,\Pi(\omega/W)$ (a rectangular spectrum of width W): this is the impulse response of an ideal low-pass filter with cutoff $W/2$.
3. $\Lambda(\omega/2W) = \Pi(\omega/2W) * \Pi(\omega/2W)$ (scaled). By the convolution theorem in frequency, $x(t) = \left(\frac{W}{\pi}\,\mathrm{sinc}(Wt)\right)^2 = \frac{W^2}{\pi^2}\,\mathrm{sinc}^2(Wt)$.
4. Applying duality to $g(t) \overset{\mathcal{F}}{\longleftrightarrow} G(j\omega)$ gives $G(jt) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi g(-\omega)$. Since $g = G(jt)$: $g(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi g(-\omega)$. By Parseval's theorem: $E_g = \frac{1}{2\pi}\int |G|^2 d\omega = \frac{1}{2\pi}\cdot 4\pi^2 E_g$, which forces $E_g = 0$ or $E_g = \infty$ unless $2\pi = 1$ — so non-trivial self-dual signals in L^2 require careful normalization (e.g. $e^{-t^2/2}$ is normalizable).

$$\boxed{e^{-|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2}{1+\omega^2}, \quad x(t) = \frac{W^2}{\pi^2}\,\mathrm{sinc}^2(Wt), \quad |G(j\omega)|^2 = 2\pi|g(\omega)|^2}$$

Problem 5

Two bandlimited signals $m_1(t)$ and $m_2(t)$, each with bandwidth W, are transmitted simultaneously using *quadrature amplitude modulation*:

$$s(t) = m_1(t)\cos(\omega_c t) - m_2(t)\sin(\omega_c t), \qquad \omega_c > W.$$

1. Derive $S(j\omega)$ in terms of $M_1(j\omega)$ and $M_2(j\omega)$.
2. Describe a complete coherent demodulation scheme that recovers $m_1(t)$ and $m_2(t)$ *separately* from $s(t)$. Prove using the CTFT that there is zero cross-channel interference.
3. The channel introduces a phase error: the received signal is $r(t) = s(t)$ but the local carrier has a phase offset ϕ, so the demodulator uses $2\cos(\omega_c t + \phi)$ and $-2\sin(\omega_c t + \phi)$. Find the recovered signals in terms of m_1, m_2, and ϕ. For what value of ϕ are the two channels completely swapped?
4. Verify using Parseval's theorem and the result of part (a) that the total transmitted average power satisfies $P_s = \frac{1}{2}(P_{m_1} + P_{m_2})$.

Solution:

1. Expanding $\cos(\omega_c t) = \frac{1}{2}(e^{j\omega_c t} + e^{-j\omega_c t})$ and similarly for sin:
$$S(j\omega) = \frac{1}{2}\big[M_1(j(\omega-\omega_c)) + M_1(j(\omega+\omega_c))\big] + \frac{j}{2}\big[M_2(j(\omega-\omega_c)) - M_2(j(\omega+\omega_c))\big].$$
2. Multiply $s(t)$ by $2\cos(\omega_c t)$, low-pass filter $\to m_1(t)$. Multiply $s(t)$ by $-2\sin(\omega_c t)$, low-pass filter $\to m_2(t)$. The cross terms produce M_2 shifted to $\pm 2\omega_c$, which are rejected by the LPF.
3. After LPF, channel 1 yields $m_1\cos\phi + m_2\sin\phi$ and channel 2 yields $-m_1\sin\phi + m_2\cos\phi$. The channels are *completely swapped* when $\phi = \pi/2$, giving outputs m_2 and $-m_1$ respectively.
4. $P_s = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}[m_1^2\cos^2 + m_2^2\sin^2 - 2m_1 m_2\cos\sin]\,dt = \frac{P_{m_1}}{2} + \frac{P_{m_2}}{2}$ since the cross term averages to zero.

$$\boxed{P_s = \tfrac{1}{2}(P_{m_1} + P_{m_2}), \quad \text{channels swap when } \phi = \pi/2}$$

Problem 6

Two causal LTI subsystems are described by:

$$H_1: \quad \frac{dy_1}{dt} + 3\,y_1(t) = x(t), \qquad H_2: \quad \frac{d^2y_2}{dt^2} + 2\frac{dy_2}{dt} + 5\,y_2(t) = \frac{dy_1}{dt} + y_1(t),$$

where the output of H_1 drives H_2.

1. Find $H_1(j\omega)$, $H_2(j\omega)$, and the overall frequency response $H(j\omega) = H_1(j\omega)\,H_2(j\omega)$.
2. Compute the impulse response $h(t)$ of the overall system using partial fractions.
3. Classify $H(j\omega)$ as low-pass, high-pass, or band-pass by evaluating $|H(j\omega)|$ at $\omega = 0$, $\omega \to \infty$, and the peak frequency ω^*.
4. The system is now changed to the following: $H_{\text{fb}}(j\omega) = H(j\omega)/[1 + H(j\omega)]$. Find the poles of $H_{\text{fb}}(j\omega)$ and determine whether the system is BIBO stable. (This is a negative feedback loop that has a unity gain. You will learn more about this in your Feedback Control Systems Course :))
5. For the input $x(t) = \cos(2t)\,u(t)$, find the *steady-state* component of $y_2(t)$ as $t \to \infty$ using the frequency response directly.

Solution:

1. $H_1(j\omega) = \dfrac{1}{j\omega + 3}$, $H_2(j\omega) = \dfrac{j\omega + 1}{(j\omega)^2 + 2(j\omega) + 5}$.
$$H(j\omega) = \frac{j\omega + 1}{(j\omega + 3)\big[(j\omega)^2 + 2(j\omega) + 5\big]}.$$
2. PFD: $H(j\omega) = \dfrac{A}{j\omega + 3} + \dfrac{Bj\omega + C}{(j\omega)^2 + 2(j\omega) + 5}$. Solving: $A = \frac{1}{5}$, $B = -\frac{1}{5}$, $C = \frac{2}{5}$.
$$h(t) = \tfrac{1}{5}e^{-3t}u(t) + e^{-t}\big[-\tfrac{1}{5}\cos(2t) + \tfrac{1}{10}\sin(2t)\big]u(t).$$
3. $|H(0)| = 1/15 > 0$; $|H(j\omega)| \to 0$ as $\omega \to \infty$ (degree 1 vs. 3). A local maximum exists between 0 and ∞: **band-pass** behaviour.
4. $1 + H(j\omega) = 0$ gives $(j\omega + 3)[(j\omega)^2 + 2(j\omega) + 5] + (j\omega + 1) = 0$. Expanding: $(j\omega)^3 + 5(j\omega)^2 + 11(j\omega) + 16 = 0$. Roots checked by Routh criterion all have negative real parts $\Rightarrow$ closed-loop is **BIBO stable**.
5. $H(j2) = \dfrac{j2 + 1}{(j2 + 3)[(j2)^2 + 2(j2) + 5]} = \dfrac{1 + 2j}{(3 + 2j)(1 + 4j)}$. $|H(j2)| = \dfrac{\sqrt{5}}{\sqrt{13}\cdot\sqrt{17}}$, $\angle H(j2) = \arctan 2 - \arctan\frac{2}{3} - \arctan 4$.
$$y_{2,\text{ss}}(t) = |H(j2)|\cos(2t + \angle H(j2)).$$

$$\boxed{H(j\omega) = \frac{j\omega + 1}{(j\omega + 3)[(j\omega)^2 + 2(j\omega) + 5]}, \quad \text{band-pass}, \quad \text{closed-loop BIBO stable}}$$

Problem 7

Define the signal
$$p(t) = \frac{\sin(Wt)}{\pi t} \cdot \frac{\sin(Bt)}{\pi t}, \qquad 0 < B < W.$$

1. Using the convolution theorem and the duality principle, find $P(j\omega)$ — the CTFT of $p(t)$ — *without computing any integral.* Sketch $P(j\omega)$ and identify it as a named standard spectral shape.
2. $p(t)$ is used as the impulse response of a filter applied to the modulated signal $s(t) = m(t)\cos(\omega_c t)$, where $m(t)$ is bandlimited to W and $\omega_c \gg W$. Assuming $\omega_c - W > B$, find the CTFT of the output $y(t) = (p * s)(t)$ and describe the filtering action in plain terms.
3. Consider the LCC differential equation
$$\frac{d^2y}{dt^2} + W^2 y(t) = p(t).$$
Take the CTFT of both sides and find $Y(j\omega)$. Identify any frequency components of $P(j\omega)$ that coincide with the natural frequency of the system and explain what phenomenon arises.

4. Apply the duality principle to $P(j\omega)$ found in part (a) to determine the CTFT of the time-domain signal $P(jt)$. Express the result as a closed-form signal and verify consistency with part (a).

Solution:

1. $\frac{\sin(Wt)}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} \Pi(\omega/2W)$ (rect of width $2W$, height 1) and similarly for B. Multiplication in time $\leftrightarrow$ convolution in frequency (scaled by $\frac{1}{2\pi}$):

$$P(j\omega) = \frac{1}{2\pi}\Pi\Big(\frac{\omega}{2W}\Big) * \Pi\Big(\frac{\omega}{2B}\Big) = \frac{1}{2\pi}\Lambda_{\text{trap}}(\omega),$$

a **trapezoidal** spectrum: flat of height $\frac{2B}{2\pi}$ for $|\omega| \le W - B$, linearly falling to zero at $|\omega| = W + B$.

2. $Y(j\omega) = P(j\omega) \cdot S(j\omega)$. Since P is a LPF-like baseband shape and $s(t)$ is bandpass around $\pm\omega_c$, with $\omega_c > W + B$, $Y(j\omega) = 0$: the filter *completely rejects* the modulated signal. If $\omega_c < W + B$, partial overlap occurs and the output is the trapezoidally windowed sideband of $m(t)$.

3. CTFT of the DE: $(-\omega^2 + W^2)Y(j\omega) = P(j\omega)$, so $Y(j\omega) = \dfrac{P(j\omega)}{W^2 - \omega^2}$. Since $P(j \pm W) \neq 0$, the denominator vanishes at $\omega = \pm W$: **resonance** occurs — the forced response is unbounded (requires an impulsive Y at $\pm W$ in the distributional sense).

4. Duality applied to $p(t) \overset{\mathcal{F}}{\longleftrightarrow} P(j\omega)$ gives $P(jt) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\, p(-\omega) = 2\pi\, p(\omega)$ (since p is even). Thus the CTFT of the trapezoidal pulse $P(jt)$ is $2\pi p(\omega)$, a product of two sinc functions — consistent with part (a) by the duality symmetry.

$$\boxed{P(j\omega) = \text{trapezoid, width } 2(W+B), \quad \text{resonance at } |\omega| = W, \quad \mathcal{F}\{P(jt)\} = 2\pi\, p(\omega) = \frac{2\pi \sin(W\omega)\sin(B\omega)}{\pi^2\omega^2}}$$

— End of Problem Set —