LUMS
A Not-for-Profit University

School of Science and Engineering

EE 310 Signals and Systems

PROBLEM SET 8

Key Concepts of Sampling, Zero-Order Hold, and the z-Transform

This problem set covers the core ideas from ideal sampling, Nyquist rate, signal reconstruction, zero-order hold, processing continuous-time signals using discrete-time systems, and the z-transform. The aim is to build both computational skill and conceptual understanding.

Ideal Sampling and Nyquist Rate

A continuous-time signal $x_c(t)$ is sampled every T_s seconds to form the discrete-time signal

$$x[n] = x_c(nT_s)$$

where the sampling frequency is

$$f_s = \frac{1}{T_s}$$

If the highest analog frequency present in the signal is $f_{\max}$, then to avoid aliasing we require

$$f_s \geq 2f_{\max}$$

The quantity $2f_{\max}$ is called the **Nyquist rate**, and the corresponding maximum sampling interval is called the **Nyquist interval**.

Ideal Reconstruction

If sampling is done above the Nyquist rate and the signal is bandlimited, then the original continuous-time signal can be perfectly reconstructed using an ideal low-pass filter, or equivalently by sinc interpolation:

$$x_c(t) = \sum_{n=-\infty}^{\infty} x[n] \, \text{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

Zero-Order Hold

A zero-order hold (ZOH) converts a discrete-time sequence into a continuous-time piecewise-constant waveform by holding each sample value constant over one sampling interval. Its impulse response is

$$h_0(t) = u(t) - u(t - T_s)$$

and its Fourier transform is

$$H_0(j\omega) = \frac{1 - e^{-j\omega T_s}}{j\omega} = T_s \, e^{-j\omega T_s/2} \, \text{sinc}\left(\frac{\omega T_s}{2\pi}\right)$$

The magnitude response is $|H_0(j\omega)| = T_s \left|\text{sinc}\left(\frac{\omega T_s}{2\pi}\right)\right|$, which introduces a frequency-dependent distortion (droop) that must be compensated if faithful reconstruction is required.

z-Transform

The bilateral z-transform of a sequence $x[n]$ is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

The region of convergence (ROC) is the set of values of z for which this sum converges.

Some standard pairs are:

$$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - az^{-1}}, \qquad |z| > |a|$$

$$-a^n u[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - az^{-1}}, \qquad |z| < |a|$$

The ROC is essential because the same algebraic expression can correspond to different time-domain signals depending on the ROC.

Problem 1

A continuous-time signal is

$$x_c(t) = \cos(200\pi t) + 2\cos(600\pi t)$$

(a) Find the frequencies present in $x_c(t)$ in Hz and determine the Nyquist rate.

(b) State whether aliasing occurs if the signal is sampled at $f_s = 1000$ Hz and at $f_s = 500$ Hz.

(c) For $f_s = 1000$ Hz, find the discrete-time sequence $x[n]$.

Problem 2

A continuous-time signal

$$x_c(t) = \cos(2\pi \cdot 700\, t)$$

is sampled at

$$f_s = 800 \text{ Hz}$$

Find the aliased frequency that appears after sampling, and write the sampled sequence $x[n]$.

Problem 3

A bandlimited continuous-time signal has highest frequency component equal to 2 kHz. It is sampled at 5 kHz.

(a) Can the signal be perfectly reconstructed in theory?

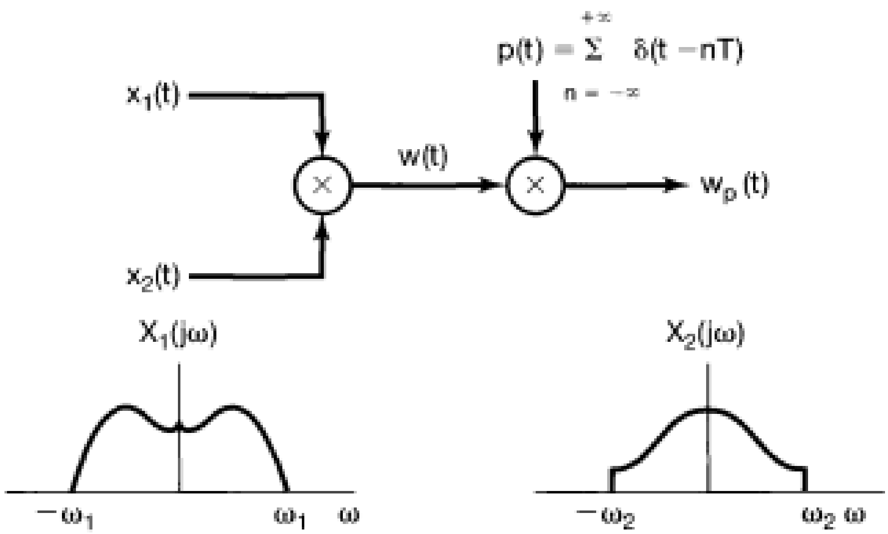
(b) Write the ideal interpolation formula for reconstruction.

(c) Briefly explain why sinc functions appear in ideal reconstruction.

Problem 4

In the system shown below, two functions of time, $x_1(t)$ and $x_2(t)$, are multiplied together, and the product $w(t)$ is sampled by a periodic impulse train. $x_1(t)$ is band limited to ω_1, and $x_2(t)$ is band limited to ω_2; that is,

$$X_1(j\omega) = 0, \quad |\omega| \geq \omega_1, \qquad X_2(j\omega) = 0, \quad |\omega| \geq \omega_2.$$

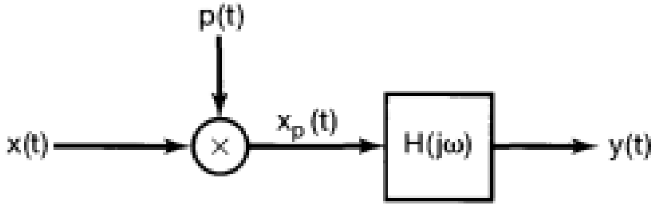


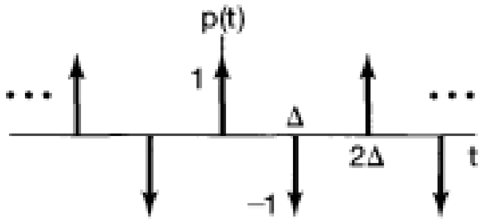
Determine the **maximum** sampling interval T such that $w(t)$ is recoverable from $w_p(t)$ through the use of an ideal lowpass filter.

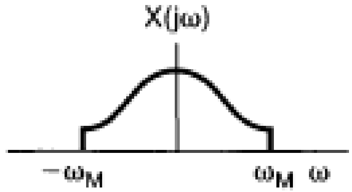
Hint: Multiplication in time = convolution in frequency. Two signals walk into a convolution... they leave wider. Bandwidth adds.

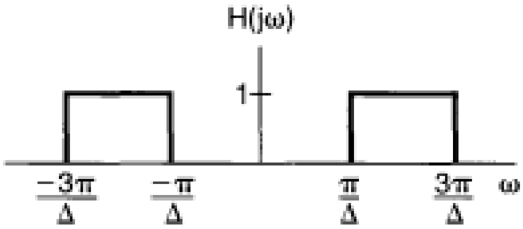
Problem 5

Shown below is a system in which the sampling signal $p(t)$ is an impulse train with alternating sign. The Fourier transform of the input signal is as indicated in the figure.









(a) For $\Delta < \pi/(2\omega_M)$, sketch the Fourier transform of $x_p(t)$ and $y(t)$.

(b) For $\Delta < \pi/(2\omega_M)$, determine a system that will recover $x(t)$ from $x_p(t)$.

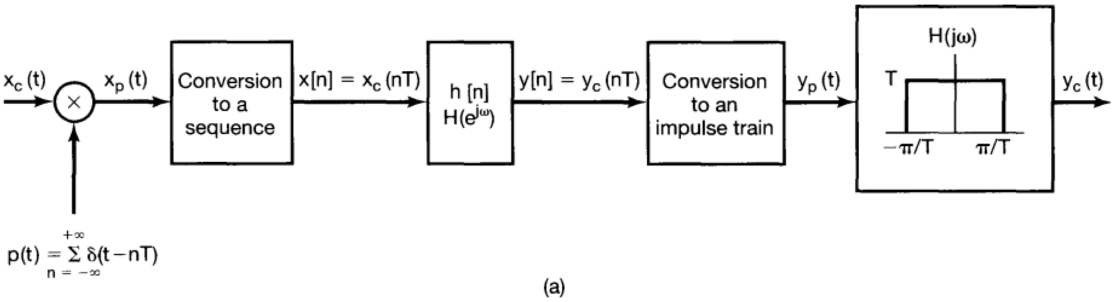
(c) For $\Delta < \pi/(2\omega_M)$, determine a system that will recover $x(t)$ from $y(t)$.

(d) What is the **maximum** value of Δ in relation to ω_M for which $x(t)$ can be recovered from either $x_p(t)$ or $y(t)$?

Hint: $+1, -1, +1, -1, \ldots = e^{j\pi n}$*. Your spectrum didn't disappear — it rage quit DC and moved to* $\pm\pi/\Delta$*. Bring it back.*

Problem 6

The figure below shows the overall system for filtering a continuous-time signal using a discrete-time filter. If $X_c(j\omega)$ and $H(e^{j\omega})$ are as shown, with $1/T = 20$ kHz, sketch $X_p(j\omega)$, $X(e^{j\omega})$, $Y(e^{j\omega})$, $Y_p(j\omega)$, and $Y_c(j\omega)$.



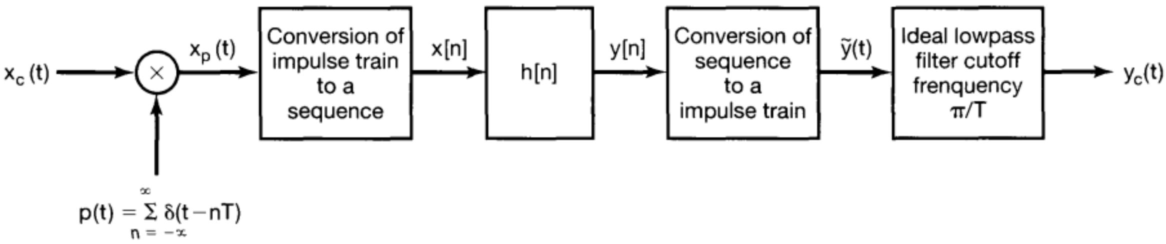
(a)

Problem 7

Shown below is a system that processes continuous-time signals using a digital filter $h[n]$ that is linear and causal with difference equation

$$y[n] = \tfrac{1}{2}\, y[n-1] + x[n].$$

For input signals that are band limited such that $X_c(j\omega) = 0$ for $|\omega| > \pi/T$, the system in the figure is equivalent to a continuous-time LTI system.



Determine the frequency response $H_c(j\omega)$ of the equivalent overall system with input $x_c(t)$ and output $y_c(t)$.

Problem 8

A discrete-time LTI system has transfer function

$$H(z) = \frac{1}{1 - 0.5z^{-1}}$$

Find the impulse response $h[n]$ and state whether the system is stable.

Problem 9

Find the bilateral z-transform and ROC of

$$x[n] = a^n u[n]$$

Also explain what changes if we instead use the unilateral z-transform.

Problem 10

Find the z-transform and ROC of

$$x[n] = -a^n u[-n-1]$$

Problem 11

Find the inverse z-transform of

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}$$

for each of the following ROCs:

(a) $|z| > \frac{1}{2}$

(b) $|z| < \frac{1}{2}$

Hint: Same formula, different ROC = different personality. One is causal and chill. The other lives in the past.

Problem 12

A system has transfer function

$$H(z) = \frac{1}{(1 - 0.2z^{-1})(1 - 0.8z^{-1})}$$

Determine the poles, the ROC for a causal system, and whether the causal system is stable.

— End of Problem Set —