LAHORE UNIVERSITY OF MANAGEMENT SCIENCES
Department of Electrical Engineering

EE310 Signals and Systems
Quiz 3 Solutions

Name: ______________________

Campus ID: ______________________

Total Marks: 10
Time Duration: 20 minutes

Question 1 (7 marks)

Let

$$x(t) = (t-1)\big[u(t-1) - u(t-2)\big]$$

be a ramp signal defined over the interval $1 \le t \le 2$, and

$$h(t) = u(t+1) - u(t-1)$$

be a rectangular pulse defined over $-1 \le t \le 1$.

(a) [1 mark] Clearly sketch $x(t)$ and $h(t)$.

(b) [5 marks] Compute the convolution

$$y(t) = x(t) * h(t)$$

by carefully determining the limits of integration.

(c) [1 mark] Sketch and label the final output $y(t)$.

Solution:

The convolution is defined as

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau.$$

The signal $x(\tau)$ is nonzero for

$$1 \le \tau \le 2,$$

and $h(t-\tau)$ is nonzero when

$$-1 \le t - \tau \le 1 \quad \Rightarrow \quad t-1 \le \tau \le t+1.$$

Thus, the limits of integration are

$$\max(1, t-1) \le \tau \le \min(2, t+1).$$

For the intervals where overlap exists, we evaluate $x(\tau) = \tau - 1$. The total duration of the non-zero output is $(1+2) = 3$, spanning $0 \le t \le 3$. We analyze the three distinct regions of overlap:

Case 1: $0 \le t < 1$
Here, $t-1 < 0$ (outside left) and $t+1 < 2$ (inside right). The limits are 1 to $t+1$.

$$y(t) = \int_1^{t+1} (\tau - 1)\, d\tau = \left[\frac{(\tau-1)^2}{2}\right]_1^{t+1} = \frac{(t+1-1)^2}{2} - 0 = \frac{t^2}{2}.$$

Case 2: $1 \le t < 2$
Here, the window fully covers the signal. The limits are 1 to 2.

$$y(t) = \int_1^{2} (\tau - 1)\, d\tau = \left[\frac{(\tau-1)^2}{2}\right]_1^{2} = \frac{1}{2} - 0 = 0.5.$$

Case 3: $2 \le t < 3$
Here, $t-1 > 1$ (inside left) and $t+1 > 2$ (outside right). The limits are $t-1$ to 2.

$$y(t) = \int_{t-1}^{2} (\tau - 1)\, d\tau = \left[\frac{(\tau-1)^2}{2}\right]_{t-1}^{2} = \frac{1}{2} - \frac{((t-1)-1)^2}{2} = \frac{1}{2} - \frac{(t-2)^2}{2}.$$

Combining all cases, the final output is:

$$y(t) = \begin{cases} 0, & t < 0, \\ \frac{t^2}{2}, & 0 \le t < 1, \\ 0.5, & 1 \le t < 2, \\ 0.5 - \frac{(t-2)^2}{2}, & 2 \le t < 3, \\ 0, & t \ge 3. \end{cases}$$

Question 2 (3 marks)

Consider two continuous-time LTI systems connected such that the output of the first is the input of second system. The first system has an impulse response

$$h_1(t) = e^{-t} u(t).$$

The second system is a differentiator with input $(x(t))$ and output $(y(t))$ related by

$$y(t) = \frac{d}{dt}\big(x(t)\big)$$

Fine the output of the second system if the input to the first system is a unit step function. (Hint: Convolution is commutative.)

Solution:

The output of the cascade is given by

$$y(t) = x(t) * h_1(t) * h_2(t).$$

Since convolution of LTI systems is associative and commutative, the order of convolution may be rearranged:

$$y(t) = \big(x(t) * h_2(t)\big) * h_1(t).$$

Now,

$$x(t) * h_2(t) = u(t) * \frac{d}{dt}\delta(t) = \frac{d}{dt} u(t) = \delta(t).$$

Therefore,

$$y(t) = \delta(t) * h_1(t) = h_1(t).$$

Hence, the overall output is

$$\boxed{y(t) = e^{-t} u(t)}.$$