LAHORE UNIVERSITY OF MANAGEMENT SCIENCES
Department of Electrical Engineering

EE310 Signals and Systems
Quiz 5 Solutions

Name: ______________________

Campus ID: ______________________

Total Marks: 10
Time Duration: 20 minutes

Question 1 (2 marks)

The **duality property** of the CTFT states: if

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega),$$

then

$$X(jt) \xleftrightarrow{\mathcal{F}} 2\pi\, x(-\omega).$$

Prove the duality property.

Solution: Proof:

Start from the inverse CTFT:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

Replace t with $-t$:

$$x(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{-j\omega t}\, d\omega$$

Now swap the variable names $t \leftrightarrow \omega$ (both range over all reals, so the equation remains valid):

$$x(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jt)\, e^{-j\omega t}\, dt$$

Rearranging:

$$2\pi\, x(-\omega) = \int_{-\infty}^{\infty} X(jt)\, e^{-j\omega t}\, dt = \mathcal{F}\{X(jt)\}$$

The right-hand side is precisely the CTFT analysis equation applied to the signal $X(jt)$. Therefore:

$$X(jt) \xleftrightarrow{\mathcal{F}} 2\pi\, x(-\omega) \qquad \blacksquare$$

Question 2 (8 marks)

Consider the signal $x(t) = e^{-a|t|}$ for constant $a > 0$.

(a) [3 marks] Compute the CTFT $X(j\omega)$ directly from the analysis equation

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt.$$

(b) [3 marks] Using the result of part (a) and the duality property proved in Question 1, determine the CTFT of

$$g(t) = \frac{1}{a^2 + t^2}.$$

(c) [2 marks] Using **Parseval's theorem** applied to $g(t)$, evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{1}{\left(a^2 + t^2\right)^2}\, dt.$$

Express your answer in terms of a.

Solution:

Part (a):

Split at $t = 0$, using $|t| = -t$ for $t < 0$ and $|t| = t$ for $t > 0$:

$$X(j\omega) = \int_{-\infty}^{0} e^{at}\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt = \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

Both integrals converge since $a > 0$:

$$= \frac{1}{a - j\omega} + \frac{1}{a + j\omega} = \frac{(a + j\omega) + (a - j\omega)}{a^2 + \omega^2}$$

$$\boxed{X(j\omega) = \frac{2a}{a^2 + \omega^2}}$$

Part (b):

We have $e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} \dfrac{2a}{a^2 + \omega^2}$.

By duality, $X(jt) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\, x(-\omega)$.

Here $X(jt) = \dfrac{2a}{a^2 + t^2} = 2a \cdot g(t)$, and $x(-\omega) = e^{-a|-\omega|} = e^{-a|\omega|}$.

Applying duality:

$$\mathcal{F}\left\{\frac{2a}{a^2 + t^2}\right\} = 2\pi\, e^{-a|\omega|}$$

Dividing both sides by $2a$:

$$\boxed{G(j\omega) = \mathcal{F}\left\{\frac{1}{a^2 + t^2}\right\} = \frac{\pi}{a}\, e^{-a|\omega|}}$$

Part (c):

Parseval's theorem states $\displaystyle\int_{-\infty}^{\infty} |g(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega)|^2\, d\omega$.

Left side:

$$\int_{-\infty}^{\infty} \frac{1}{(a^2 + t^2)^2}\, dt = I$$

Right side:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\pi}{a}\right)^2 e^{-2a|\omega|}\, d\omega = \frac{\pi^2}{2\pi a^2} \cdot 2 \int_{0}^{\infty} e^{-2a\omega}\, d\omega = \frac{\pi}{a^2} \cdot \frac{1}{2a} = \frac{\pi}{2a^3}$$

Therefore:

$$\boxed{I = \frac{\pi}{2a^3}}$$