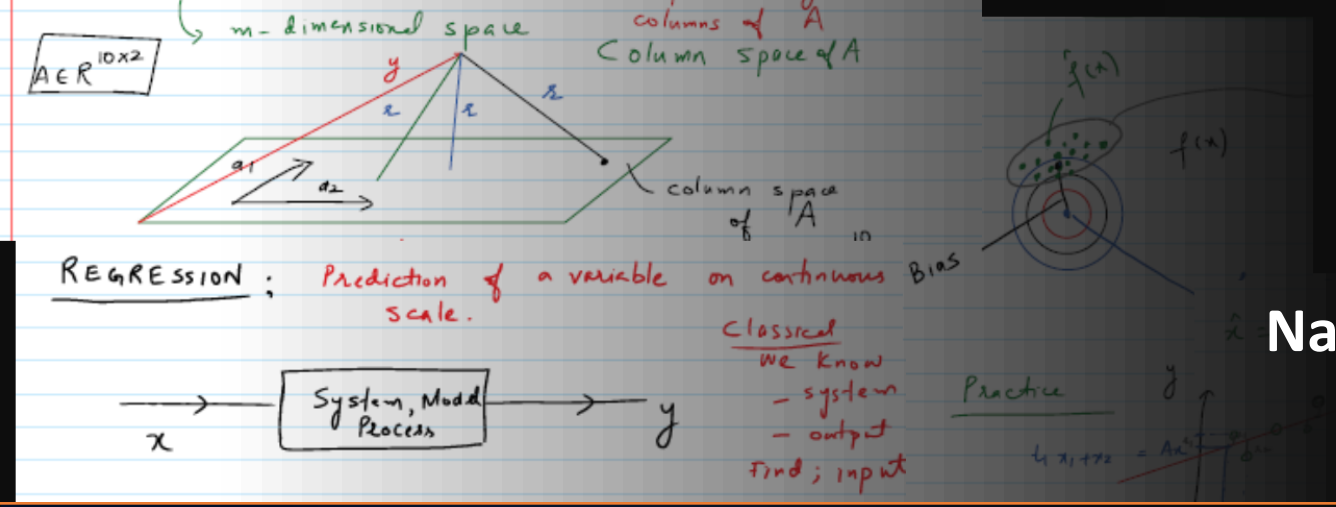

Machine Learning
EE514 – CS535

Naïve Bayes Classifier and Bayesian Network Introduction

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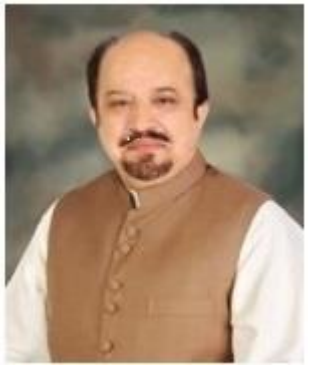
Outline

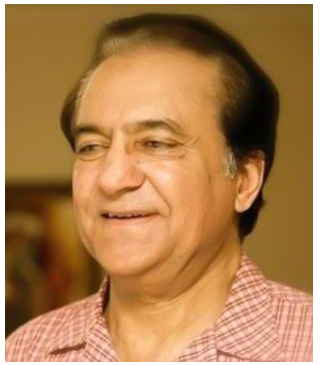
- Naïve Bayes Classifier
- Introduction to Bayesian Network

Reference: Chapter 6 (Machine Learning by Tom Mitchell)

Naïve Bayes Classifier

Example:





- Given the name of a person, we want to predict the sex of a person.
- For example, we have a person say 'Firdous'.
- Classifying 'Firdous' as female or male is equivalent to asking is it more probable that 'Firdous' is male or female.
- Mathematically, which one is greater $P(\text{male} \mid \text{Firdous})$ or $P(\text{female} \mid \text{Firdous})$
- Let's apply Bayes theorem

$$P(\text{male} \mid \text{Firdous}) = \frac{P(\text{Firdous} \mid \text{male})P(\text{male})}{P(\text{Firdous})}$$

Probability of being named Firdous given male

Probability of being male

Probability of being named Firdous

Naïve Bayes Classifier

Example:

- We will look at the database of names vs gender.

$$P(\text{male} \mid \text{Firdous}) = \frac{P(\text{Firdous} \mid \text{male})P(\text{male})}{P(\text{Firdous})}$$

- $n = 48$, male count $= 28$, female count $= 20$
- Firdous: male count $=4$, female count $= 6$

$$P(\text{Firdous} \mid \text{male}) = \frac{4}{28} \qquad P(\text{male}) = \frac{28}{48}$$

$$P(\text{Firdous}) = \frac{10}{48}$$

$$P(\text{male} \mid \text{Firdous}) = 0.4$$

Name	Gender
Ahtesham	male
Iyad	male
Maleeha	female
Firdous	male
Shawal	male
Firdous	male
Ahmed	male
Zainab	female
Firdous	female
Ubaid	male
Badar	male
Firdous	female
Hassan	male
Kash	male
Hajira	female
Shehla	female
Firdous	female
Haram	female
Abdullah	male
Fahad	male

Bilal	male
Habeel	male
Farhan	male
Firdous	male
Anam	female
Firdous	female
Osama	male
Fatima	female
Mahnoor	female
Balaj	male
Razi	male
Zuhaib	male
Firdous	female
Shaharyar	male
Firdous	female
Ali	male
Mustansar	male
Sana	female
Anam	female
Marium	female
Khadija	female
Salaar	male
Faaiq	male
Hamza	male
Mahad	male
Ayesha	female
Firdous	male
Jawaria	female

Naïve Bayes Classifier

Example:

- *Given Outlook, Temperature, Humidity and Wind Information, we want to carry out prediction for Play: Yes or No.*

- Mathematically, which one is greater

$$P(\text{Play} = \text{Yes} \mid \text{Outlook},\ \text{Temp.},\ \text{Humidity},\ \text{Wind})$$

$$P(\text{Play} = \text{No} \mid \text{Outlook},\ \text{Temp.},\ \text{Humidity},\ \text{Wind})$$

- Predict for Sunny outlook, High humidity, Cool temperatue and Weak wind.

- Predict the most likely.

Day	Outlook	Temp.	Humidity	Wind	Play
D1	Sunny	Hot	High	Weak	No
D2	Sunny	Hot	High	Strong	No
D3	Overcast	Hot	High	Weak	Yes
D4	Rain	Mild	High	Weak	Yes
D5	Rain	Cool	Normal	Weak	Yes
D6	Rain	Cool	Normal	Strong	No
D7	Overcast	Cool	Normal	Strong	Yes
D8	Sunny	Mild	High	Weak	No
D9	Sunny	Cool	Normal	Weak	Yes
D10	Rain	Mild	Normal	Weak	Yes
D11	Sunny	Mild	Normal	Strong	Yes
D12	Overcast	Mild	High	Strong	Yes
D13	Overcast	Hot	Normal	Weak	Yes
D14	Rain	Mild	High	Strong	No

Reference: Section 6.9.1 (Machine Learning by Tom Mitchell)

Naïve Bayes Classifier

Example:

$$P(\text{Play} = \text{Yes} \mid \text{Outlook} = \text{Sunny},\ \text{Temp} = \text{Cool},\ \text{Humidity} = \text{High},\ \text{Wind} = \text{Weak})$$

$$= \frac{P(\text{Outlook} = \text{Sunny},\ \text{Temp} = \text{Cool},\ \text{Humidity} = \text{High},\ \text{Wind} = \text{Strong} \mid \text{Play} = \text{Yes})\, P(\text{Play} = \text{Yes})}{P(\text{Outlook} = \text{Sunny},\ \text{Temp} = \text{Cool},\ \text{Humidity} = \text{High},\ \text{Wind} = \text{Strong})}$$

Naïve Assumption:

- Feature are mutually independent given the label!

$$P(\text{Outlook} = \text{Sunny},\ \text{Temp} = \text{Cool},\ \text{Humidity} = \text{High},\ \text{Wind} = \text{Strong} \mid \text{Play} = \text{Yes})$$

$$= P(\text{Outlook} = \text{Sunny} \mid \text{Play} = \text{Yes})\, P(\text{Temp} = \text{Cool} \mid \text{Play} = \text{Yes})\, P(\text{Humidity} = \text{High} \mid \text{Play} = \text{Yes})\, P(\text{Wind} = \text{Strong} \mid \text{Play} = \text{Yes})$$

Naïve Bayes Classifier

Example:

$$P(\text{Outlook} = \text{Sunny} \mid \text{Play} = \text{Yes}) = \frac{2}{9}$$

$$P(\text{Temp} = \text{Cool} \mid \text{Play} = \text{Yes}) = \frac{3}{9}$$

$$P(\text{Humidity} = \text{High} \mid \text{Play} = \text{Yes}) = \frac{3}{9}$$

$$P(\text{Wind} = \text{Strong} \mid \text{Play} = \text{Yes}) = \frac{3}{9}$$

$$P(\text{Play} = \text{Yes}) = \frac{9}{14}$$

$$P(\text{Outlook} = \text{Sunny} \mid \text{Play} = \text{No}) = \frac{3}{5}$$

$$P(\text{Temp} = \text{Cool} \mid \text{Play} = \text{No}) = \frac{1}{5}$$

$$P(\text{Humidity} = \text{High} \mid \text{Play} = \text{No}) = \frac{4}{5}$$

$$P(\text{Wind} = \text{Strong} \mid \text{Play} = \text{No}) = \frac{3}{5}$$

$$P(\text{Play} = \text{No}) = \frac{5}{14}$$

Day	Outlook	Temp.	Humidity	Wind	Play
D1	Sunny	Hot	High	Weak	No
D2	Sunny	Hot	High	Strong	No
D3	Overcast	Hot	High	Weak	Yes
D4	Rain	Mild	High	Weak	Yes
D5	Rain	Cool	Normal	Weak	Yes
D6	Rain	Cool	Normal	Strong	No
D7	Overcast	Cool	Normal	Strong	Yes
D8	Sunny	Mild	High	Weak	No
D9	Sunny	Cool	Normal	Weak	Yes
D10	Rain	Mild	Normal	Weak	Yes
D11	Sunny	Mild	Normal	Strong	Yes
D12	Overcast	Mild	High	Strong	Yes
D13	Overcast	Hot	Normal	Weak	Yes
D14	Rain	Mild	High	Strong	No

Naïve Bayes Classifier

Example:

$$P(\text{Outlook} = \text{Sunny} \mid \text{Play} = \text{Yes})\ P(\text{Temp} = \text{Cool} \mid \text{Play} = \text{Yes})\ P(\text{Humidity} = \text{High} \mid \text{Play} = \text{Yes})\ P(\text{Wind} = \text{Strong} \mid \text{Play} = \text{Yes})$$

$$\times P(\text{Play} = \text{Yes}) \qquad = \frac{2}{9} \times \frac{3}{9} \times \frac{3}{9} \times \frac{3}{9} \times \frac{9}{14} \qquad = 0.0053$$

$$P(\text{Outlook} = \text{Sunny} \mid \text{Play} = \text{No})\ P(\text{Temp} = \text{Cool} \mid \text{Play} = \text{No})\ P(\text{Humidity} = \text{High} \mid \text{Play} = \text{No})\ P(\text{Wind} = \text{Strong} \mid \text{Play} = \text{No})$$

$$\times P(\text{Play} = \text{No}) \qquad = \frac{3}{5} \times \frac{1}{5} \times \frac{4}{5} \times \frac{3}{5} \times \frac{5}{14} \qquad = 0.0206$$

$$P(\text{Play} = \text{Yes} \mid \text{Outlook} = \text{Sunny},\ \text{Temp} = \text{Cool},\ \text{Humidity} = \text{High},\ \text{Wind} = \text{Strong}) = \frac{0.0053}{0.0053 + 0.0206} = 0.2046$$

$$P(\text{Play} = \text{No} \mid \text{Outlook} = \text{Sunny},\ \text{Temp} = \text{Cool},\ \text{Humidity} = \text{High},\ \text{Wind} = \text{Strong}) = \frac{0.0206}{53 + 0.0206} = 0.7954$$

Play = No *is more likely!*

Naïve Bayes Classifier

Generative Classifier:

- Attempts to model class, that is, build a generative statistical model that informs us how a given class would generate input data.

- Ideally, we want to learn the joint distribution of the input **x** and output label y, that is, P(**x**,y).

- For a test-point, generative classifiers predict which class would have **most-likely** generated the given observation.

- Mathematically, prediction for input **x** is carried out by computing the conditional probability P(y|**x**) and selecting the most-likely label y.

- Using the Bayes rule, we can compute P(y|**x**) by computing P(y) and P(**x**|y).

- Estimating $P(y)$ and $P(\mathbf{x}|y)$ is called generative learning.

Naïve Bayes Classifier

Overview of Naïve Bayes Classifier:

- We have $D = \{(\mathbf{x_1}, y_1), (\mathbf{x_2}, y_2), \ldots, (\mathbf{x_n}, y_n)\} \subseteq \mathcal{X}^d \times \mathcal{Y}$

  $\mathcal{Y} = \{1, 2, \ldots, M\}$ (M-class classification)

Key Idea:

- Estimate $P(y|\mathbf{x})$ from the data using the Bayes Theorem.
- Using Bayes theorem and MAP learning framework, we can write this as

$$h_{\text{MAP}}(\mathbf{x}) = \underset{y \in \mathcal{Y}}{\text{maximize}} \quad P(y \mid \mathbf{x}) = \underset{y \in \mathcal{Y}}{\text{maximize}} \quad \frac{P(\mathbf{x} \mid y)\, P(y)}{P(\mathbf{x})} = \underset{y \in \mathcal{Y}}{\text{maximize}} \quad P(\mathbf{x} \mid y)\, P(y)$$

- Estimating $P(y)$ is easy. If y takes on discrete binary values, coin tossing or spam vs non-spam for example, we simply need to count how many times we observe each class outcome.
- Estimating $P(\mathbf{x}|y)$, however, is not easy, Why?

Naïve Bayes Classifier

Overview of Naïve Bayes Classifier:

Example:

- $M = 2$ and features $d = 6$. Assuming binary features/classification.
- We want to estimate

$$P(\mathbf{x} \mid y) = P(x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}, x^{(5)}, x^{(6)} | y)$$

- How many parameters do we need to fully estimate $P(\mathbf{x}|y)$?
- We need to represent all 2^6 outcomes or probabilities for each $y = 0, 1$.
- For d binary features, we need to represent all 2^d outcomes.
- Learning the values for the full conditional probability would require enormous amounts of data.

time	Inputv1	Inputv2	Inputv3	Inputv4	Inputv5	Inputv6	output
19:50:00	1	0	0	1	0	0	1
19:55:00	1	0	0	1	0	0	0
20:00:00	1	0	0	1	0	0	1
20:05:00	1	1	1	0	0	0	1
20:10:00	1	1	1	0	0	0	1
20:15:00	1	1	0	1	0	0	1
20:20:00	1	1	0	1	1	0	0
20:25:00	1	0	0	1	1	0	1
20:30:00	1	0	0	1	1	0	1
20:35:00	0	0	0	1	0	0	1
20:40:00	1	0	0	1	1	0	1
20:45:00	0	0	0	1	0	0	0

Naïve Bayes Classifier

Naïve Bayes Classifier:

- To overcome this requirement of enormous data for the computation of conditional probability, we can make a 'naive Bayes' assumption.

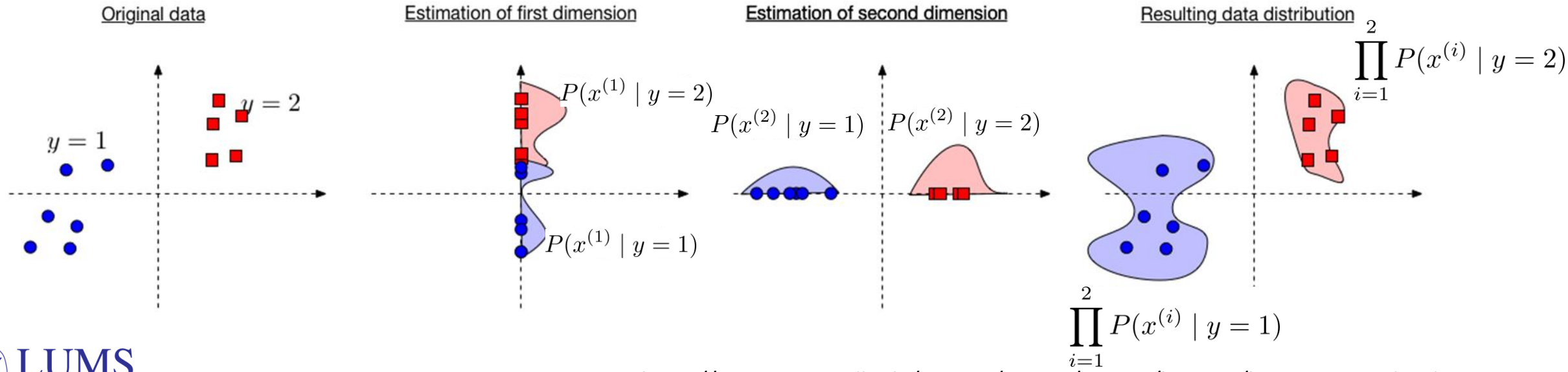
Naïve Assumption:

- Features are mutually independent given the label!
- Consequence: $P(\mathbf{x} \mid y) = P(x^{(1)}, x^{(2)}, \ldots, x^{(d)}|y) = \prod_{i=1}^{d} P(x^{(i)} \mid y)$
- How many probabilities now?

  one for each feature/label.

  $2d$

Interpretation[1]:



1. Source: https://www.cs.cornell.edu/courses/cs4780/2018sp/lectures/lecturenote05.html

Naïve Bayes Classifier

Naïve Bayes Classifier:

- We can reformulate our hypothesis function, referred to as Naive Bayes (NB) Classifier, as

$$h_{\text{NB}}(\mathbf{x}) = \underset{y \in \mathcal{Y}}{\text{maximize}} \quad P(y \mid \mathbf{x}) = \underset{y \in \mathcal{Y}}{\text{maximize}} \quad \prod_{i=1}^{d} P(x^{(i)} \mid y)\, P(y)$$

- Maximizes the log (natural, ln) of the function instead.

$$h_{\text{NB}}(\mathbf{x}) = \underset{y \in \mathcal{Y}}{\text{maximize}} \quad \sum_{i=1}^{d} \log \left(P(x^{(i)} \mid y)\, P(y) \right)$$

$$= \underset{y \in \mathcal{Y}}{\text{maximize}} \quad \sum_{i=1}^{d} \log P(x^{(i)} \mid y) \; + \; \log P(y)$$

- How many probabilities?

$2d + 1$

Naïve Bayes Classifier

Naïve Bayes Classifier - Training:

Assume each feature and label as a binary variable

- Hypothesis space: $2d + 1$ different binomial distributions.
  - $P(x^{(i)} \mid y)$ and $P(y)$ for each $x^{(i)}$ and each $y = \{0, 1\}$, $i = 1, 2, \ldots, d$.
  - Each probability can be parameteterized by a single variable θ.
- We treat learning of each of these as a separate MLE problem.

$$P(x^{(i)} = j \mid y = k) = \frac{\text{count}(x^{(i)} = j \text{ and } y = k)}{\text{count}(y = k)}, \quad j, k \in \{0, 1\}$$

$$P(y = k) = \frac{\text{count}(y = k)}{\text{count}(y = 0) + \text{count}(y = 1)} = \frac{\text{count}(y = k)}{n}, \quad k \in \{0, 1\}$$

- We compute these probabilities during training stage.
- As we saw earlier, these probability estimates maximizes the likelihood.

Naïve Bayes Classifier

Naïve Bayes Classifier - Prediction:

Assume each feature and label as a binary variable

- For a new test-point $\mathbf{x}_{\text{new}}$, we assign the label as

$$h_{\text{NB}}(\mathbf{x}_{\text{new}}) = \underset{y \in \mathcal{Y}}{\text{maximize}} \quad P(y \mid \mathbf{x}_{\text{new}}) = \underset{y \in \mathcal{Y}}{\text{maximize}} \quad \prod_{i=1}^{d} P(x_{\text{new}}^{(i)} \mid y)\, P(y)$$

We have a problem here!

- We have a product of probabilities. If any of the estimated probability is zero, the product would be zero.

Solution: Additive Smoothing or Laplace Smoothing

$$P(x^{(i)} = j \mid y = k) = \frac{\text{count}(x^{(i)} = j \text{ and } y = k) + \ell}{\text{count}(y = k) + \ell R}, \quad j, k \in \{0, 1\}$$

$$P(y = k) = \frac{\text{count}(y = k) + \ell}{n + \ell M}, \quad k \in \{0, 1\}$$

- Here $\ell > 0$. If $\ell = 1$, we refer to it as add-1 smoothing.
- R is the number of values $x^{(i)}$ can take. For binary case, $R = 2$.
- M is the number of classes. For binary case $M = 2$.

Naïve Bayes Classifier

Naïve Bayes Classifier - Extensions:

- We have $D = \{(\mathbf{x_1}, y_1), (\mathbf{x_2}, y_2), \ldots, (\mathbf{x_n}, y_n)\} \subseteq \mathcal{X}^d \times \mathcal{Y}$

  $\mathcal{Y} = \{1, 2, \ldots, M\}$ (M-class classification)

- We assume that each feature $x^{(i)}$ takes L_i values, that is, $x^{(i)} \in \{1, 2, \ldots, L_i\}$.

How many probability tables do we have if we have d features and M labels?

- $dM + 1$: we have one probability table for each feature and each value of the label and one more table for the prior $P(y)$.
- The set of tables for a single feature (for all labels y) is referred to as a conditional probability table (CPT), and here we have d of those.

Incorporating model parameters in the formulation

- We considered a binary case and assumed that a single parameter characterizes probability model associated with each feature.
- In general, we can have parameters defining the probability model and we learn parameters of the probability model during the learning stage.

Naïve Bayes Classifier

Naïve Bayes Classifier – Extensions:

Gaussian Naïve Bayes – Continuous Features:

- In practice, some features are discrete (e.g., gender, marital status) and some are continuous (weight).
- The probability model or distribution for each $x^{(i)}$ can be parameterized differently.
- If $x^{(i)} \in \mathbf{R}$, what kind of distribution can we use for $P(x^{(i)}|y)$?
- For real-valued features, we often use a Gaussian distribution to **model probability density function**, that is,

$$p(x^{(i)} \mid y = k) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x^{(i)} - \mu)^2}{2\sigma^2}\right) \qquad p(x^{(i)} \mid y = k) \sim N(\mu, \sigma^2).$$

- For succinct representation, the depenence of μ and σ on feature index i and label index k is dropped. We can have different distributions or parameters for each i and each k. just like we have different probabilities for discrete features.

Naïve Bayes Classifier

Naïve Bayes Classifier – Extensions:

Gaussian Naïve Bayes – Training:

- We have $p(x^{(i)} \mid y = k) \sim N(\mu, \sigma^2)$, given data we want to learn μ and σ for each i and each k.
- Given i and k, we compute the μ and σ as sample mean and sample variance, where the sample corresponds to $x^{(i)}$ for which associated label $y = k$.

$$\mu = \frac{1}{\text{count}(y = k)} \sum_{j=1}^{n} \delta(y_j - k)\ x_j^{(i)}$$

$$\sigma^2 = \frac{1}{\text{count}(y = k)} \sum_{j=1}^{n} \delta(y_j - k)\ \left(x_j^{(i)} - \mu\right)^2$$

- For each label y, we need to estimate d means and d variances during training.

Naïve Bayes Classifier

Naïve Bayes Classifier - Summary:

- In Naïve Bayes, we compute the probabilities or parameters of the distribution defining probabilities and use these to carry out predictions.
- Naïve Bayes can handle missing values by ignoring the sample during probability computation, is robust to outliers and irrelevant features.
- Naïve Bayes algorithm is very easy to implement for applications involving textual information data (e.g., sentiment analysis, news article classification, spam filtering).
- Convergence is quicker relative to logistic regression (to be studied later) that discriminative in nature.
- It performs well even when the independence between features assumption does not hold.
- The resulting decision boundaries can be non-linear and/or piecewise.
- Disadvantage: It is not robust to redundant features. If the features have a strong relationship or correlation with each other, Naïve Bayes is not a good choice. Naïve Bayes has high bais and low variance and there are no regularization here to adjust the bias thing

NB Classifier – Text Classification

Text Classification Overview:

- Applications of text classification include
  - Sentiment analysis
  - Spam detection
  - Language Identification; to name a few.

Classification Problem:

Input: a document and a fixed set of classes (e.g., spam, non-spam)

Output: a predicted class for the document

Classification Methods:

- **Hand-coded rules:** Rules based on combinations of words or other features
  - e.g., spam: black-list-address OR ("dollars" AND "you have been selected")
  - Accuracy can be high if rules carefully refined by **expert**
  - But building and **maintaining** these rules is **expensive**

Ref: "Speech and Language Processing" by *Daniel Jurafsky & James H. Martin*
Source: https://web.stanford.edu/~jurafsky/slp3/

NB Classifier – Text Classification

Text Classification – Supervised Learning:

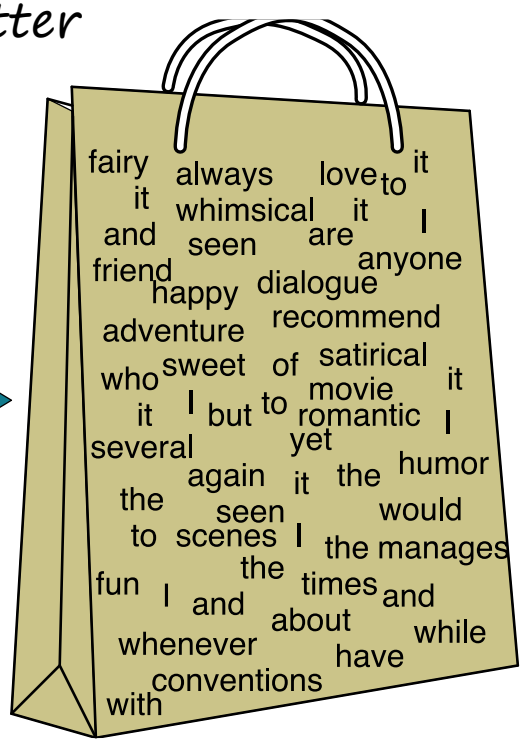
Input: *a document and a fixed set of classes (e.g., spam, non-spam)*
+ training data (n labeled documents)

Output: *a predicted class for the document*

Bag of Words – Representation of a document for classification:

Assumption: *Position doesn't matter*

I love this movie! It's sweet, but with satirical humor. The dialogue is great and the adventure scenes are fun... It manages to be whimsical and romantic while laughing at the conventions of the fairy tale genre. I would recommend it to just about anyone. I've seen it several times, and I'm always happy to see it again whenever I have a friend who hasn't seen it yet!



it	6
I	5
the	4
to	3
and	3
seen	2
yet	1
would	1
whimsical	1
times	1
sweet	1
satirical	1
adventure	1
genre	1

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Ref: "Speech and Language Processing" by *Daniel Jurafsky & James H. Martin*
Source: https://web.stanford.edu/~jurafsky/slp3/

NB Classifier – Text Classification

Text Classification – Terminology and Preprocessing :

- Corpus: A collection of documents; data.

- Vocabulary, denoted by V, is the union of all the word types in all classes (not just one class).

Preprocessing documents:

- Clean the corpus: (e.g., Hello, hello or hello! should be considered the same)
  - Remove numbers, punctuation and excessive white spaces
  - Use lowercase representation

- Stop words concept: very frequent words (**a** or **the**)
  - Sort vocabulary with respect to frequency, call the top 5 or 20 words the stopword list and remove from all of the documents or from the vocabulary.

- In naïve Bayes, it's more common to **not** remove stop words and use all the words.

- After pre-processing, create a **mega document** for each class by concatenating all the documents of the class.

- Use bag of words on mega document to obtain a frequency table for each class.

Ref: "Speech and Language Processing" by *Daniel Jurafsky & James H. Martin*
Source: https://web.stanford.edu/~jurafsky/slp3/

NB Classifier – Spam Filtering

Example: Spam vs Non-Spam:

Category	Document
Spam	send us your password
Spam	review us
Spam	send us your account
Spam	send your password
Non-spam	password review
Non-spam	send us your review
?	review us now
?	review account

Issue 1:

'**now**' is not in the training data.

- unknown word or out of vocabulary word.

Solution:

remove out of vocabulary word from the test document.

Issue 2:

'**account**' is only available in one class

Solution:

Use add-1 smoothing. We will see this shortly.

- Vocabulary, V = {send, us, your, password, review, account}

NB Classifier – Spam Filtering

Naïve Bayes (NB) Classification:

- NB Classifier:

$$h_{\text{NB}}(\mathbf{x}) = \underset{y \in \mathcal{Y}}{\text{maximize}} \quad P(y \mid \mathbf{x}) = \underset{y \in \mathcal{Y}}{\text{maximize}} \quad \prod_{i=1}^{d} P(x^{(i)} \mid y)\, P(y)$$

- $\mathbf{x}$ represents the test document for which we want to carry out prediction. Each feature represents a word in the document.
- d here represents the number of words in the test document.

- For $\mathbf{x}$ = "review us now", $d = 3$.

- For $\mathbf{x}$ = "review account", $d = 2$.

Ref: "Speech and Language Processing" by *Daniel Jurafsky & James H. Martin*
Source: https://web.stanford.edu/~jurafsky/slp3/

NB Classifier – Spam Filtering

Naïve Bayes (NB) Classification – Example:

Category	Document
Spam	send us your password
Spam	review us
Spam	send us your account
Spam	send your password
Non-spam	password review
Non-spam	send us your review
?	review us now
?	review account

Bag of Words

Vocabulary	Spam Count	Non-spam Count
send	3	1
us	3	1
your	3	1
password	2	1
review	1	2
account	1	0
	13	6

- For $\mathbf{x} =$"review us now", $d = 3$.

We compute $P(\text{Spam} \mid \mathbf{x})$ and $P(\text{Non} - \text{spam} \mid \mathbf{x})$

NB Classifier – Spam Filtering

Naïve Bayes (NB) Classification – Example:

- For $\mathbf{x}$ = "review us now".
- Ignore 'now': unknown word, out of vocabulary
- We compute $P(\mathbf{x} \mid \text{Spam})\, P(\text{Spam})$ and $P(\mathbf{x} \mid \text{Non}-\text{spam})\, P(\text{Non}-\text{spam})$

$$P(\mathbf{x} \mid \text{Spam})\, P(\text{Spam}) = P(\text{review} \mid \text{Spam}) P(\text{us} \mid \text{Spam})\, P(\text{Spam})$$

$$P(\text{review} \mid \text{Spam}) = \frac{1}{13} \qquad P(\text{us} \mid \text{Spam}) = \frac{3}{13} \qquad P(\text{Spam}) = \frac{4}{6}$$

$$P(\mathbf{x} \mid \text{Spam})\, P(\text{Spam}) = 0.012$$

$$P(\text{review} \mid \text{Non}-\text{spam}) = \frac{2}{6} \quad P(\text{us} \mid \text{Non}-\text{spam}) = \frac{1}{6} \quad P(\text{Non}-\text{spam}) = \frac{2}{6}$$

$$P(\mathbf{x} \mid \text{Non}-\text{spam})\, P(\text{Non}-\text{spam}) = 0.0185$$

Vocabulary	Spam Count	Non-spam Count
send	3	1
us	3	1
your	3	1
password	2	1
review	1	2
account	1	0
	13	6

Document is likely a non-spam.

NB Classifier – Spam Filtering

Naïve Bayes (NB) Classification – Example:

- For $\mathbf{x}$ = "review account".
- For 'account': non-spam count is zero. Consequently, $P(\text{account} \mid \text{Non} - \text{spam}) = 0$.

Solution: Add 1 smoothing

$$P(\text{Spam}) = \frac{4}{6} \qquad P(\text{Non} - \text{spam}) = \frac{2}{6}$$

$$P(\text{review} \mid \text{Spam}) = \frac{1+1}{13+6} = \frac{2}{19} \qquad P(\text{account} \mid \text{Spam}) = \frac{1+1}{13+6} = \frac{2}{19}$$

We have added numerator factor times the size of the vocabulary in the denominator.

$$P(\text{review} \mid \text{Non} - \text{spam}) = \frac{2+1}{6+6} = \frac{3}{12} \qquad P(\text{account} \mid \text{Non} - \text{spam}) = \frac{0+1}{6+6} = \frac{1}{12}$$

$$P(\mathbf{x} \mid \text{Spam})\, P(\text{Spam}) = 0.00738$$

$$P(\mathbf{x} \mid \text{Non} - \text{spam})\, P(\text{Non} - \text{spam}) = 0.00694$$

Document is likely a spam.

Vocabulary	Spam Count	Non-spam Count
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us	3	1
your	3	1
password	2	1
review	1	2
account	1	0
	13	6

Outline

- Bayesian Learning Framework
  - MAP Estimation
  - ML Estimation
- Linear Regression as Maximum Likelihood Estimation
- Naïve Bayes Classifier
- Introduction to Bayesian Network

Bayesian Networks Introduction

Overview:

- Using Bayes theorem, we developed the following classifier:

$$h_{\text{MAP}}(\mathbf{x}) = \underset{y \in \mathcal{Y}}{\text{maximize}} \quad P(y \mid \mathbf{x}) = \underset{y \in \mathcal{Y}}{\text{maximize}} \quad \frac{P(\mathbf{x} \mid y)\, P(y)}{P(\mathbf{x})} = \underset{y \in \mathcal{Y}}{\text{maximize}} \quad P(\mathbf{x} \mid y)\, P(y)$$

- Estimation/computation of $P(\mathbf{x} \mid y)$ requires enormous amounts of data.

- We simplified using naïve Bayes assumption: features are independent.

(Too simple to hold!)

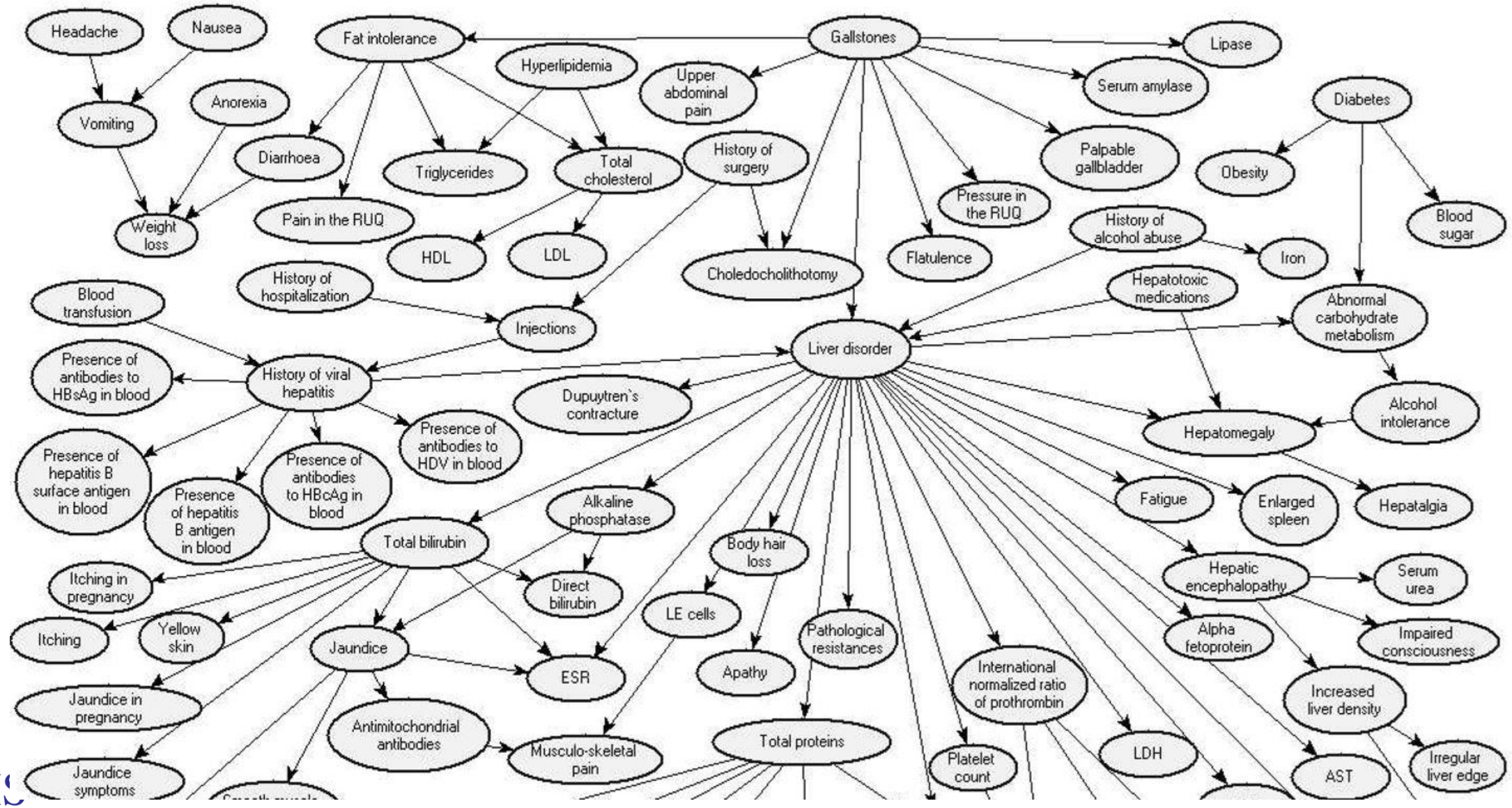
- Generalizes **the idea of naïve Bayes** to **model distributions over groups of variables** with more complex conditional independence relationships.

- Bayesian network – a graphical model for representing probabilistic relationships among inputs, labels.

- **Idea:** A Bayesian network consists of **a collection of conditional probability distributions** such that their **product** is a **full joint distribution** over all the variables.

Bayesian Networks Introduction

Overview – Example: Bayesian Network for Liver Disorder Diagnosis :



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Source: "A Bayesian Network Model for Diagnosis of Liver Disorders" –A. Onisko, M. Druzdzel, and H. Wasyluk, Sept. 1999.

Bayesian Networks Introduction

Introduction:

- Bayesian Network: Directed Acyclic Graph (DAG) + Conditional Probability Tables or Distributions (CPT or CPD).

- Bayesian networks can be visualized by drawing a directed graph (nodes + edges).

- We represent variables in the form of **nodes**.

- These nodes can be labels or features: we **do not** make any distinction between features and labels during training as they are all treated the same way.

- Edges or arcs represent the relationship or dependence between the variables.

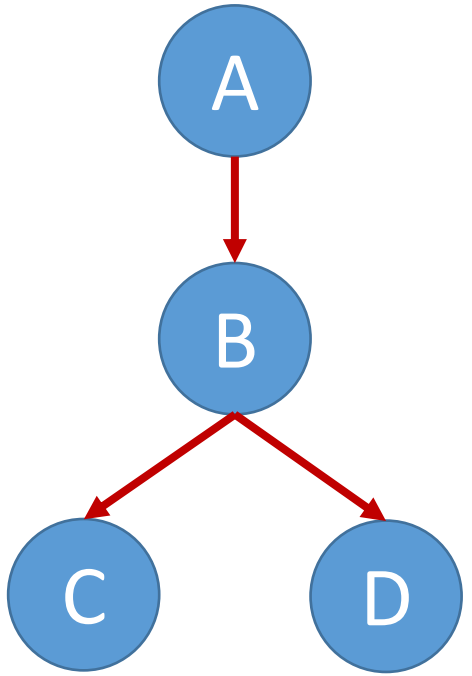
- Nodes and edges represent the conditional independence relationships between the variables.

- We **may** also represent causality in the Bayesian network.
  - Causality means the effect of one variable on the other.
  - Incorporating causality can help us defining a structured graph.

Bayesian Networks Introduction

Example:

- Bayesian Network: DAG + CPT
- Node: represents a random variable
- Directed Edge
  - B is a parent of C and D
  - Direction indicates the causation
- Assuming each variable is Bernoulli RV.



CPT for each node:
Each node has a conditional probability table that quantifies the relationship with the parent node.

A	P(A)
0	0.6
1	0.4

A	B	P(B|A)
0	0	0.01
0	1	0.99
1	0	0.7
1	1	0.3

B	C	P(C|B)
0	0	0.4
0	1	0.6
1	0	0.9
1	1	0.1

B	D	P(D|B)
0	0	0.01
0	1	0.99
1	0	0.05
1	1	0.95

Bayesian Networks Introduction

Example:

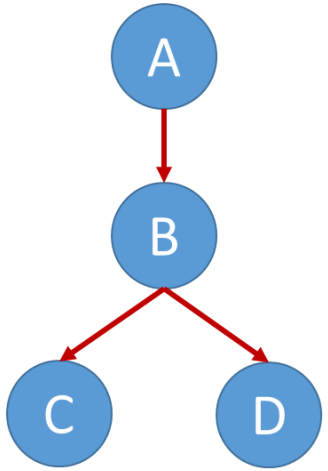
- For this network, we want to compute the following joint distribution:

$$P(A=1,B=1,C=1,D=1) = P(A=1) \times P(B=1,C=1,D=1 \mid A=1)$$

Exploiting independence between C and D,
and conditional independence between C (or D) and A

$$= P(A=1) \times P(B=1 \mid A=1) \times P(C=1 \mid B=1) \times P(D=1 \mid B=1)$$

$$= 0.4 \times 0.3 \times 0.1 \times 0.95 = 0.0114$$



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Bayesian Networks Introduction

Formulation:

- For variables $X_1, X_2, \ldots, X_d$, exploiting network structure, we can write

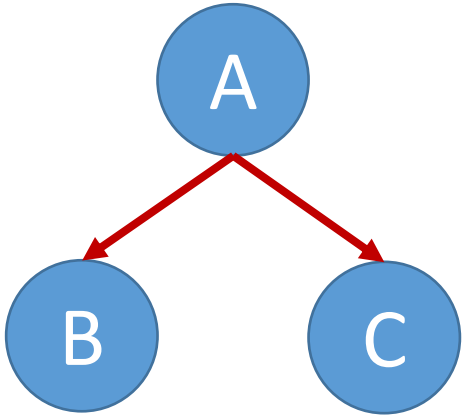
$$P(X_1, X_2, \ldots, X_d) = \prod_i P(X_i | \text{parents}(X_i))$$

- Using Bayesian network, we have a structured and compact representation of the joint distribution.

Independences:

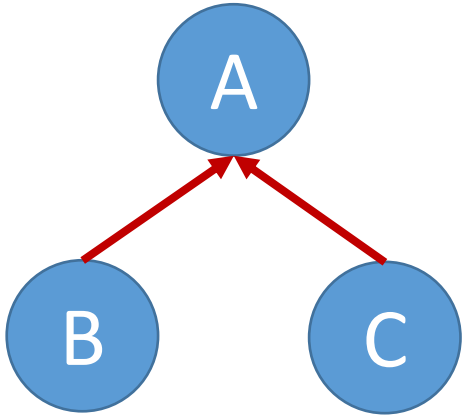
- Marginal independence:

$$P(A, B, C) = P(A)P(B)P(C)$$



- Conditionally independent effects

$$P(A, B, C) = P(B|A)P(C|A)P(A)$$



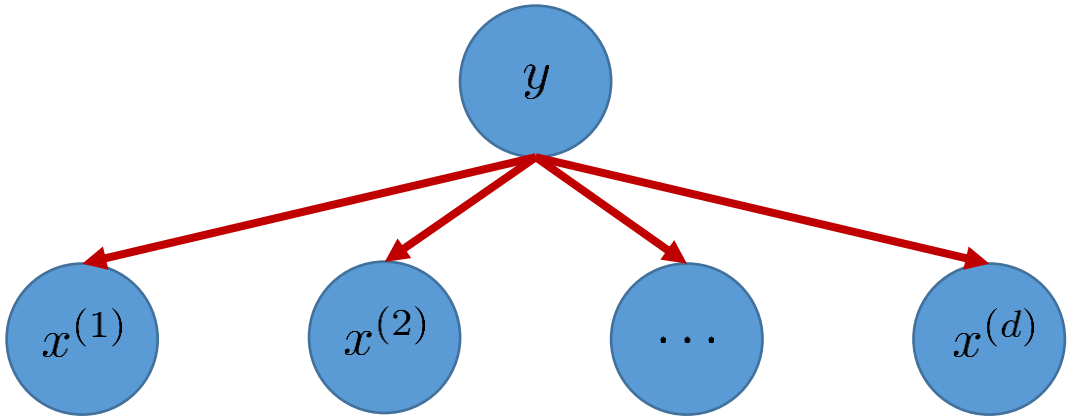
- Independent causes

$$P(A, B, C) = P(A|B, C)P(B)P(C)$$

Bayesian Networks Introduction

Naïve Bayes Network (Classifier):

- If $x^{(1)}, x^{(2)}, \ldots, x^{(d)}$ represent the features and y is a label of the class.



$$P(y \mid \mathbf{x}) = \prod_{i=1}^{d} P(x^{(i)} \mid y)\, P(y)$$

Bayesian Networks Introduction

Prediction or Inference using Bayesian Network:

- We compute posterior probabilities given some evidence.

- Mathematically, we want to compute $P(Y|X)$, where X represent the evidence (e.g., features) and Y is the query variable (e.g., label).

- In general, **exact** inference is intractable (NP hard).

- There are assumptions (e.g., simplest: Naïve Bayes) and approximate methods (e.g., Monte Carlo) which can be used to carry out inference efficiently.

Learning of Bayesian Network:

- Structure (nodes + edges) is given, we learn conditional probabilities using the training data.

- If structure is not given, we use domain knowledge along with the training data to learn the both the structure and conditional probabilities using the data.

Bayesian Networks Introduction

Bayesian Networks – Example: Disease Diagnosis